Universidad Carlos III de Madrid
Department of Mathematics

Ph.D. Thesis

Generalized Coherent Pairs and Sobolev Orthogonal Polynomials

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A mis padres,
a Dios, a la Virgen,
y a Iván.

Abstract

This work presents a study of (M,N)-coherent pairs of order (m,k) of sequences of orthogonal polynomials of a continuous and discrete variable on the real line and on the unit circle. This concept extends all the generalizations of the notion of, in our terminology, $(1,0)$-coherent pair, studied in the literature, which was first introduced as coherent pair by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna in 1991 (see [48]).

A pair of regular (resp. weakly quasi-definite, resp. regular Hermitian) linear functionals $(\mathcal{U},\mathcal{V})$ is said to be a *(M,N)-coherent pair of order (m,k)* (resp. *(M,N)-D_ν-coherent pair of order (m,k)*, resp. *(M,N)-coherent pair of order (m,k) on the unit circle*) if their corresponding families of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ $\big($resp. $\{P_n(x)\}_{n=0}^{\Upsilon_0}$ and $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$, resp. $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}\big)$ satisfy a structure relation

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

$$\Bigg(\text{resp.} \quad \sum_{i=0}^{M} a_{i,n} D_\nu^m P_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} D_\nu^k Q_{n+k-i}(x),$$

$$0 \leq n \leq \min\{\Upsilon_0 - m, \Upsilon_1 - k\},\ 0 \leq M, m \leq \Upsilon_0,\ 0 \leq N, k \leq \Upsilon_1,\Bigg)$$

$$\Bigg(\text{resp.} \quad \sum_{i=0}^{M} a_{i,n} \phi_{n+m-i}^{(m)}(z) = \sum_{i=0}^{N} b_{i,n} \psi_{n+k-i}^{(k)}(z), \quad n \geq 0,\Bigg)$$

where M, N, m, and k are non-negative integers, $\{a_{i,n}\}_{n\geq 0}$, $0 \leq i \leq M$, and $\{b_{i,n}\}_{n\geq 0}$, $0 \leq i \leq N$, are sequences of complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ if $i > n$. Here $p^{(m)}$ denotes the mth derivative of p (resp. D_ν is either D_ω or D_q defined by

$$(D_\omega p)(x) = \frac{p(x+\omega) - p(x)}{\omega}, \quad \omega \in \mathbb{C} \setminus \{0\},$$

$$(D_q p)(x) = \frac{p(qx) - p(x)}{(q-1)x}, \quad x \neq 0, \quad (D_q p)(0) = p'(0), \quad q \in \mathbb{C} \setminus \{0,1\}).$$

When $k=0$, $(\mathcal{U},\mathcal{V})$ is called a *(M,N)-coherent pair of order m* (resp. *(M,N)-D_ν-coherent pair of order m*, resp. *(M,N)-coherent pair of order m on the unit circle*), and when $(m,k) = (1,0)$, $(\mathcal{U},\mathcal{V})$ is said to be a *(M,N)-coherent pair* (resp. *(M,N)-D_ν-coherent pair*, resp. *(M,N)-coherent pair on the unit circle*).

We prove that the semiclassical (resp. D_ν-semiclassical) character of the linear functionals $\mathcal{U}$ and $\mathcal{V}$ is a necessary condition for the (M,N)-coherence (resp. (M,N)-D_ν-coherence) condition of order (m,k) of the pair $(\mathcal{U},\mathcal{V})$, whenever $m \neq k$. Additionally, from either the (M,N) or the (M,N)-D_ν -coherence relation of order (m,k), we show that the linear functionals are related by an expression of rational type, generalizing all the results found on this topic in the literature.

On the other hand, we also generalize several recent results in the framework of Sobolev orthogonal polynomials and their connections with coherent pairs, considering the Sobolev inner products

$$\left\langle p(x), r(x)\right\rangle_\lambda = \left\langle \mathcal{U}, p(x)r(x)\right\rangle + \lambda\left\langle \mathcal{V}, p^{(m)}(x)r^{(m)}(x)\right\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$

$$\left\langle p(x), r(x)\right\rangle_{\lambda,\nu} = \left\langle \mathcal{U}, p(x)r(x)\right\rangle + \lambda\left\langle \mathcal{V}, \left(D_\nu^m p\right)(x)\left(D_\nu^m r\right)(x)\right\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$

$$\left\langle p(z), r(z)\right\rangle_\lambda = \left\langle \mathcal{U}, p(z)\overline{r}(1/z)\right\rangle + \lambda\left\langle \mathcal{V}, p^{(m)}(z)\overline{r^{(m)}}(1/z)\right\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$

where $\mathcal{U}$ and $\mathcal{V}$ constitute a (M,N)-coherent pair of order m, a (M,N)-D_ν-coherent pair of order m, and a (M,N)-coherent pair of order m on the unit circle, respectively. In particular, we show how to compute the coefficients of the Fourier expansions of functions on appropriate Sobolev spaces in terms of the sequences of Sobolev polynomials orthogonal with respect to those inner products.

Furthermore, we give additional properties for the particular cases when $(\mathcal{U},\mathcal{V})$ is either a $(1,0)$ or a $(1,1)$ (resp. a $(1,0)$-D_ν or a $(1,1)$-D_ν) -coherent pair of order m, or when one of the linear functionals in a (M,N) (resp. (M,N)-D_ν) -coherent pair of order (m,k) is classical (resp. D_ν-classical). Besides, we analyze $(1,1)$-coherent pairs on the unit circle when one of the linear functionals is either the Lebesgue or Bernstein-Szegő linear functional.

Moreover, we study the (M,N) and (M,N)-D_ν -coherence relations (resp. the (M,N)-coherence relation on the unit circle) from a matrix point of view, from which we obtain results that involve the monic Jacobi matrices (resp. the Hessenberg matrices) associated with the linear functionals in such a coherent pair.

Finally, in the Introduction we present in more detail a complete historical summary of coherent pairs as well as the original contributions of this dissertation.

Keywords: coherent pairs, Sobolev inner products, structure relations, inverse problems, difference operator D_ω, q-derivative D_q, linear functionals, regular linear functionals, weakly quasi-definite linear functionals, hermitian linear functionals, semiclassical linear functionals, Stieltjes functions, Carathéodory functions, orthogonal polynomials, discrete orthogonal polynomials, q-orthogonal polynomials, orthogonal polynomials on the unit circle, Sobolev orthogonal polynomials, classical orthogonal polynomials, Lebesgue linear functional, Bernstein-Szegő linear functional, Fourier expansions, Fourier coefficients, approximation by polynomials, algorithms, monic Jacobi matrix, Hessenberg matrix unit circle.

Resumen

Este trabajo presenta un estudio de los pares (M, N)-coherentes de orden (m, k) de sucesiones de polinomios ortogonales de una variable continua en la recta real y en la circunferencia unidad, y de una variable discreta. Este concepto extiende todas las generalizaciones de la noción de, en nuestra terminología, par (1,0)-coherente, estudiadas en la literatura, el cual fue introducido por primera vez como par coherente por A. Iserles, P. E. Koch, S. P. Nørsett, y J. M. Sanz-Serna en 1991 (véase [48]).

Un par de funcionales lineales regulares (resp. débilmente cuasi-definidos, resp. Hermitianos regulares) $(\mathcal{U}, \mathcal{V})$ se dice que es un *par* (M, N)*-coherente de orden* (m, k) (resp. *par* (M, N)*-*D_ν*-coherente de orden* (m, k), resp. *par* (M, N)*-coherente de orden* (m, k) *en la circunferencia unidad*) si sus correspondientes familias de polinomios ortogonales mónicos $\{P_n(x)\}_{n\geq 0}$ y $\{Q_n(x)\}_{n\geq 0}$ $\big($resp. $\{P_n(x)\}_{n=0}^{\Upsilon_0}$ y $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$, resp. $\{\phi_n(z)\}_{n\geq 0}$ y $\{\psi_n(z)\}_{n\geq 0}\big)$ satisfacen una relación de estructura

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

$$\Bigg(\text{resp.} \quad \sum_{i=0}^{M} a_{i,n} D_\nu^m P_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} D_\nu^k Q_{n+k-i}(x),$$

$$0 \leq n \leq \min\{\Upsilon_0 - m, \Upsilon_1 - k\},\ 0 \leq M, m \leq \Upsilon_0,\ 0 \leq N, k \leq \Upsilon_1,\Bigg)$$

$$\left(\text{resp.} \quad \sum_{i=0}^{M} a_{i,n} \phi_{n+m-i}^{(m)}(z) = \sum_{i=0}^{N} b_{i,n} \psi_{n+k-i}^{(k)}(z), \quad n \geq 0,\right)$$

donde M, N, m, y k son números enteros no negativos, $\{a_{i,n}\}_{n\geq 0}$, $0 \leq i \leq M$, y $\{b_{i,n}\}_{n\geq 0}$, $0 \leq i \leq N$, son sucesiones de números complejos tales que $a_{M,n} \neq 0$ si $n \geq M$, $b_{N,n} \neq 0$ si $n \geq N$, y $a_{i,n} = b_{i,n} = 0$ si $i > n$, y $p^{(m)}$ indica la derivada m-ésima de p (resp. D_ν es la derivada D_ω ó D_q definida por

$$(D_\omega p)(x) = \frac{p(x+\omega) - p(x)}{\omega}, \quad \omega \in \mathbb{C} \setminus \{0\},$$

$$(D_q p)(x) = \frac{p(qx) - p(x)}{(q-1)x},\ \ x \neq 0, \quad (D_q p)(0) = p'(0), \quad q \in \mathbb{C} \setminus \{0, 1\}).$$

Cuando $k = 0$, $(\mathcal{U}, \mathcal{V})$ se denomina un *par* (M, N)*-coherente de orden* m (resp. *par* (M, N)*-*D_ν*-coherente de orden* m, resp. *par* (M, N)*-coherente de orden* m *en la circunferencia*

unidad), y cuando $(m, k) = (1, 0)$, $(\mathcal{U}, \mathcal{V})$ se dice que es un *par* (M, N)*-coherente* (resp. *par* (M, N)*-*D_ν*-coherente*, resp. *par* (M, N)*-coherente en la circunferencia unidad*).

Se prueba que el carácter semiclásico (resp. D_ν-semiclásico) de los funcionales lineales $\mathcal{U}$ y $\mathcal{V}$ es una condición necesaria para la condición de (M, N)-coherencia (resp. (M, N)-D_ν-coherencia) de orden (m, k) del par $(\mathcal{U}, \mathcal{V})$, siempre que sea $m \neq k$. Adicionalmente, a partir de la relación de (M, N) ó (M, N)-D_ν -coherencia de orden (m, k), se demuestra que los funcionales lineales están relacionados por una expresión de tipo racional, generalizando todos los resultados encontrados en la literatura en este tema.

Por otro lado, también se generalizan varios resultados recientes en el marco de los polinomios ortogonales de Sobolev y sus conexiones con pares coherentes, considerando los productos internos de Sobolev

$$\begin{aligned}
\big\langle p(x), r(x)\big\rangle_\lambda &= \big\langle \mathcal{U}, p(x)r(x)\big\rangle + \lambda\Big\langle \mathcal{V}, p^{(m)}(x)r^{(m)}(x)\Big\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,\\
\big\langle p(x), r(x)\big\rangle_{\lambda,\nu} &= \big\langle \mathcal{U}, p(x)r(x)\big\rangle + \lambda\Big\langle \mathcal{V}, \big(D_\nu^m p\big)(x)\big(D_\nu^m r\big)(x)\Big\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,\\
\big\langle p(z), r(z)\big\rangle_\lambda &= \big\langle \mathcal{U}, p(z)\overline{r}(1/z)\big\rangle + \lambda\Big\langle \mathcal{V}, p^{(m)}(z)\overline{r^{(m)}}(1/z)\Big\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,
\end{aligned}$$

donde $\mathcal{U}$ y $\mathcal{V}$ constituyen un par (M, N)-coherente de orden m, un par (M, N)-D_ν-coherente de orden m, y un par (M, N)-coherente de orden m en la circunferencia unidad, respectivamente. En particular, se muestra cómo calcular los coeficientes de los desarrollos de Fourier de funciones en espacios de Sobolev apropiados en términos de las sucesiones de polinomios de Sobolev ortogonales con respecto a dichos productos internos.

Además, se dan propiedades adicionales para los casos particulares cuando $(\mathcal{U}, \mathcal{V})$ es un par $(1, 0)$ ó $(1, 1)$ (resp. $(1, 0)$-D_ν ó $(1, 1)$-D_ν) -coherente de orden m, ó cuando uno de los funcionales lienales en un par (M, N) (resp. (M, N)-D_ν) -coherente de orden (m, k) es clásico (resp. D_ν-clásico). También, se analizan los pares $(1, 1)$-coherentes en la circunferencia unidad cuando uno de los funcionales lineales es el correspondiente a las medidas de Lebesgue ó Bernstein-Szegő.

Por otra parte, se estudian las relaciones de (M, N) y (M, N)-D_ν -coherencia (resp. la relación de (M, N)-coherencia en la circunferencia unidad) desde un punto de vista matricial. De aquí se siguen resultados que involucran las matrices mónicas de Jacobi (resp. las matrices de Hessenberg) asociadas con los funcionales lineales en dicho par coherente.

Finalmente, en la Introducción se presenta una pormenorizada reseña histórica de los pares coherentes, así como las contribuciones originales de esta tesis.

Contents

Introduction

This work deals with two main topics: coherent pairs and inverse problems in the theory of Orthogonal Polynomials. Coherent Pairs emerged in the context of Sobolev orthogonal polynomials, more specifically, in the framework of a constructive theory of such orthogonal polynomials, because Sobolev inner products become more treatable from algebraic and analytical points of view when the measures involved therein form a coherent pair. On the other hand, in the study of inverse problems, the aim is to find the relation between two regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ assuming their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are related by an algebraic expression

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

where M, N, m, k are nonnegative integer numbers, $\{a_{i,n}\}_{n\geq 0}$, $0 \leq i \leq M$ and $\{b_{j,n}\}_{n\geq 0}$, $0 \leq j \leq N$, are sequences of complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$. It is called (M, N)-coherence relation of order (m, k) and $(\mathcal{U}, \mathcal{V})$ is said to be a ***(M,N)-coherent pair of order (m,k)***. When $k = 0$, we will say that $(\mathcal{U}, \mathcal{V})$ is a ***(M,N)-coherent pair of order m***, and, when $m = 1$ and $k = 0$, $(\mathcal{U}, \mathcal{V})$ is said to be a ***(M,N)-coherent pair***.

Background

1. Coherent Pairs of Measures on the Real Case and the Derivative Operator

The notion of *coherent pair* was introduced in 1991 by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna (see [48]). They stated that a pair of positive Borel measures (μ_0, μ_1) supported on the real line with finite moments of all orders is (in our terminology) a $(1, 0)$-coherent pair if and only if there exist nonzero constants $\{a_n\}_{n\geq 1}$ such that their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, respectively, satisfy

$$\frac{P'_{n+1}(x)}{n+1} + a_n \frac{P'_n(x)}{n} = Q_n(x), \quad a_n \neq 0,\, n \geq 1. \tag{0.0.1}$$

This condition of coherence arose as a sufficient condition for the existence of a relation

$$P_{n+1}(x) + \frac{n+1}{n}\, a_n P_n(x) = S_{n+1}(x;\lambda) + c_{n,\lambda} S_n(x;\lambda), \quad n \geq 1, \tag{0.0.2}$$

where $\{c_{n,\lambda}\}_{n\geq 1}$ are rational functions in $\lambda > 0$ and $\{S_n(x;\lambda)\}_{n\geq 0}$ is the SMOP associated with the Sobolev inner product

$$\langle p(x), r(x)\rangle_\lambda = \int_{-\infty}^{\infty} p(x)r(x)d\mu_0(x) + \lambda \int_{-\infty}^{\infty} p'(x)r'(x)d\mu_1(x), \quad \lambda > 0, \tag{0.0.3}$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients. Also, they studied the case when the measure μ_0 is classical (Laguerre and Jacobi). Furthermore, they introduced the notion of symmetrically coherent pair, when the two measures μ_0 and μ_1 are symmetric (i.e., invariant under the transformation $x \mapsto -x$) and the subscripts in (0.0.1) are changed appropriately. Finally, when μ_0 and μ_1 form a coherent pair (or symmetrically coherent pair), they obtained and implemented an efficient algorithm to compute the Fourier-Sobolev coefficients $\{f_n(\lambda)/s_n(\lambda)\}_{n\geq 0}$ with

$$f_n(\lambda) = \langle f(x), S_n(x;\lambda)\rangle_\lambda \quad \text{and} \quad s_n(\lambda) = \langle S_n(x;\lambda), S_n(x;\lambda)\rangle_\lambda, \quad n \geq 0, \tag{0.0.4}$$

of the Fourier expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n(\lambda)}{s_n(\lambda)}\, S_n(x;\lambda), \tag{0.0.5}$$

for a smooth function $f(x)$ in the Sobolev space

$$W^{1,2}[I,\mu_0,\mu_1] = \left\{ f : I \to \mathbb{R} \,|\, f \in L^2_{\mu_0}(I)\,,\ f' \in L^2_{\mu_1}(I) \right\}, \tag{0.0.6}$$

where I is a open interval of the real line. An important remark pointed out in [47] was that this algorithm does not need the explicit expressions of the Sobolev orthogonal polynomials $S_n(x;\lambda)$, $n \geq 0$.

Later on, in 1993, F. Marcellán, M. Alfaro, and M. L. Rezola ([8]) summarized the study of coherent pairs from another point of view, and H. G. Meijer ([97]) stated new interesting examples of coherent pairs and symmetrically coherent pairs.

Afterwards, in 1995, F. Marcellán, T. Pérez, and M. Piñar ([73]) showed that if a pair of definite positive linear functionals $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-coherent pair, then both of them are semiclassical, $\mathcal{V}$ is of class at most 1, and $\mathcal{U}$ is of class at most 6. Moreover, they proved that there exist polynomials $\varphi(x)$ and $\rho(x)$ such that $\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\varphi(x)) \leq 3$ and $\deg(\rho(x)) \leq 2$.

On the other hand, F. Marcellán and J. Petronilho ([75]) studied (0.0.1), where $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the sequences of monic polynomials orthogonal with respect to some regular linear functionals, and they solved the problem when one of the functionals is classical, i.e., Hermite, Laguerre, Jacobi (positive definite case) or Bessel (regular case).

Besides, F. Marcellán, T. E. Pérez, J. C. Petronilho, and M. A. Piñar ([74]) analyzed the coherence coefficients in (0.0.1), when either $\{P_n(x)\}_{n\geq 0}$ or $\{Q_n(x)\}_{n\geq 0}$ is a SMOP.

Finally, in 1997, in [99] (see also [98]), H. G. Meijer determined all $(1, 0)$-coherent pairs $(\mathcal{U}, \mathcal{V})$ of regular linear functionals. He proved that at least one of the linear functionals, either $\mathcal{U}$ or $\mathcal{V}$, must be classical (Laguerre or Jacobi). Moreover, he showed that there are only two cases:

- If $\mathcal{U}$ is a classical linear functional, then there exist polynomials $\sigma_{\mathcal{U}}(x)$, $\tau_{\mathcal{U}}(x)$, and $\rho_{\mathcal{U}}(x)$, with $\deg(\sigma_{\mathcal{U}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{U}}(x)) = \deg(\rho_{\mathcal{U}}(x)) = 1$, such that

$$D(\sigma_{\mathcal{U}}(x)\mathcal{U}) = \tau_{\mathcal{U}}(x)\mathcal{U} \quad \text{and} \quad \sigma_{\mathcal{U}}(x)\mathcal{U} = \rho_{\mathcal{U}}(x)\mathcal{V}.$$

- If $\mathcal{V}$ is a classical linear functional, then there exist polynomials $\sigma_{\mathcal{V}}(x)$, $\tau_{\mathcal{V}}(x)$, and $\rho_{\mathcal{V}}(x)$, with $\deg(\sigma_{\mathcal{V}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{V}}(x)) = \deg(\rho_{\mathcal{V}}(x)) = 1$, such that

$$D(\sigma_{\mathcal{V}}(x)\mathcal{V}) = \tau_{\mathcal{V}}(x)\mathcal{V} \quad \text{and} \quad \sigma_{\mathcal{V}}(x)\mathcal{U} = \rho_{\mathcal{V}}(x)\mathcal{V}.$$

Meijer also determined all symmetrically $(1, 0)$-coherent pairs, providing similar results to those obtained in the no symmetrical case. Indeed, they correspond to Hermite and Gegenbauer cases.

Later on, in 2004, A. Delgado and F. Marcellán ([32, 33]) extended the notion of a coherent pair to *generalized coherent pairs* (in our terminology, $(1, 1)$-coherent pair), studying the relation

$$\frac{P'_{n+1}(x)}{n+1} + a_n \frac{P'_n(x)}{n} = Q_n(x) + b_n Q_{n-1}(x), \quad a_n \neq 0,\ n \geq 1.$$

They proved that this is a necessary and sufficient condition for the relation (0.0.2). Also, they determined all $(1, 1)$-coherent pairs of linear functionals (b_n can be zero). They proved that at least one of the regular linear functionals, either $\mathcal{U}$ or $\mathcal{V}$, must be semiclassical of class at most 1, generalizing the results obtained by H. G. Meijer for $(1, 0)$-coherent pairs. Moreover, they showed that there are only two cases of these $(1, 1)$-coherent pairs:

- If $\mathcal{U}$ is a semiclassical linear functional given by $D(\sigma_{\mathcal{U}}(x)\mathcal{U}) = \tau_{\mathcal{U}}(x)\mathcal{U}$ with $\deg(\sigma_{\mathcal{U}}(x)) \leq 3$ and $\deg(\tau_{\mathcal{U}}(x)) \leq 2$, then there exists a constant $C_{\mathcal{U}}$ such that $\sigma_{\mathcal{U}}(x)\mathcal{U} = (x - C_{\mathcal{U}})\mathcal{V}$.

- If $\mathcal{V}$ is a semiclassical linear functional given by $D(\sigma_{\mathcal{V}}(x)\mathcal{V}) = \tau_{\mathcal{V}}(x)\mathcal{V}$ with $\deg(\sigma_{\mathcal{V}}(x)) \leq 3$ and $\deg(\tau_{\mathcal{V}}(x)) \leq 2$, then there exists a constant $C_{\mathcal{V}}$ such that $\sigma_{\mathcal{V}}(x)\mathcal{U} = (x - C_{\mathcal{V}})\,\mathcal{V}$.

Finally, a generalization of this situation to symmetrically coherent pairs is stated by A. Delgado and F. Marcellán in [32, 34], in 2005.

In 2001, another generalization of coherent pair was introduced by F. Marcellán, A. Martínez-Finkelshtein, and J. Moreno-Balcázar (see [71]). A pair of positive measures supported on the real line (μ_0, μ_1) is said to be a κ-*coherent pair* (with our notation, a $(\kappa+1, 0)$-coherent pair), $\kappa \geq 0$, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$\frac{P'_{n+1}(x)}{n+1} + \sum_{j=1}^{\kappa+1} a_{j,n} \frac{P'_{n-j+1}(x)}{n-j+1} = Q_n(x), \quad a_{\kappa+1,n} \neq 0,\, n \geq \kappa+1.$$

Some nontrivial examples of κ-coherent pairs were studied as well as the following relation between Sobolev polynomials $\{S_n(x;\lambda)\}_{n\geq 0}$ associated with the inner product (0.0.3) and the polynomials $\{P_n(x)\}_{n\geq 0}$ associated with the first measure of this product was stated as a necessary condition for $(\kappa+1, 0)$-coherence relation

$$P_{n+1}(x) + \sum_{j=1}^{\kappa+1} \frac{n+1}{n-j+1}\, a_{j,n} P_{n-j+1}(x) = S_{n+1}(x;\lambda) + \sum_{j=1}^{\kappa+1} c_{j,n,\lambda} S_{n-j+1}(x;\lambda), \quad n \geq \kappa+1. \tag{0.0.7}$$

Notice that, we get (0.0.2) when $\kappa = 0$.

In this way, in 1995, M. G. de Bruin and H. G. Meijer ([27]) introduced the notion of *generalized coherent pair*, meaning κ-coherent pair with $\kappa = 1$ (for us $(2,0)$-coherent pair), for positive Borel measures with finite moments. In this work the relation (0.0.7) for $\kappa = 1$ was obtained and the case $\mu_0 = \mu_1 = \mu$ was analyzed. They concluded that the structure relation of generalized coherence is a sufficient condition for the classical character of μ. This characterization for classical polynomials also was proved by F. Marcellán, A. Branquinho, and J. Petronilho in 1994 ([70]). They showed that the classical SMOP are the only SMOP such that each polynomial of the sequence is a linear combination of the derivatives of at most three consecutive polynomials of the same family, i.e, a regular linear functional $\mathcal{U}$ is classical if and only if $(\mathcal{U}, \mathcal{U})$ is a $(2,0)$-coherent pair.

In 2001, this concept of generalized coherent pair ($(2,0)$-coherent pair) was studied by K. H. Kwon, J. H. Lee, and F. Marcellán ([66]) for regular linear functionals. They concluded that if $(\mathcal{U}, \mathcal{V})$ is a generalized coherent pair of linear functionals, then $\mathcal{U}$ and $\mathcal{V}$ must be semiclassical, $\mathcal{U}$ of class at most 6 and $\mathcal{V}$ of class at most 2. They also studied the case when either $\mathcal{U}$ or $\mathcal{V}$ is classical. Additionally, they gave an efficient algorithm for computing the Sobolev-Fourier coefficients $\{f_n(\lambda)/s_n(\lambda)\}_{n\geq 0}$ (see (0.0.4)-(0.0.6)) when (μ_0, μ_1) is a generalized coherent pair of positive Borel measures with finite moments (based on the algebraic relations obtained by M. G. de Bruin and H. G. Meijer in [27]).

They generalized the algorithm proved by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna in [48], and again, they do not need the explicit expressions of the Sobolev orthogonal polynomials $\{S_n(x;\lambda)\}_{n\geq 0}$.

In 1999 and 2000, P. Maroni and R. Sfaxi ([88] and [90]) introduced the notion of *coherent pair associated with* $\phi(x)$, a monic polynomial of degree t, *with index* s, $s \geq 0$. In such a case, a pair $(\{Q_n(x)\}_{n\geq 0}, \{P_n(x)\}_{n\geq 0})$ of SMOP with respect to the pair of regular linear functionals $(\mathcal{V}, \mathcal{U})$ satisfies

$$\phi(x)Q_n(x) = \sum_{j=n-s}^{n+t} a_{n,j} \frac{P'_{j+1}(x)}{j+1}, \quad a_{n,n-s} \neq 0, \;\; n \geq s\,. \tag{0.0.8}$$

If $Q_n(x) = P_n(x)$ for all $n \geq 0$, $\{P_n(x)\}_{n\geq 0}$ is said to be a *diagonal sequence associated with* $\phi(x)$ *with index* s. They obtained necessary and sufficient conditions for (0.0.8) as well as some results for the dual bases of $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$. Furthermore, they concluded that the diagonal sequences are semiclassical. In 2006, J. Alaya and R. Sfaxi ([2]) continued this study. In 2007, F. Marcellán and R. Sfaxi ([82]) established a (second) structure relation to characterize semiclassical orthogonal polynomials as follows. $\mathcal{U}$ is a semiclassical linear functional of class at most N if and only if $(\mathcal{U}, \mathcal{U})$ is a $(N+r, 2N)$-coherent pair. Here the non-negative integer r is the degree of one of the polynomials involved in the Pearson equation satisfied by $\mathcal{U}$.

Notice that from the three-term recurrence relation that $\{Q_n(x)\}_{n\geq 0}$ satisfies and expressing $\phi(x)Q_n(x)$ as a linear combination of polynomials $Q_{n-t}, \ldots, Q_{n+t}$, we get

$$\frac{P'_{n+t+1}(x)}{n+t+1} + \sum_{j=n-s}^{n+t-1} a_{n,j} \frac{P'_{j+1}(x)}{j+1} = Q_{n+t}(x) + \sum_{j=n-t}^{n+t-1} b_{n,j} Q_j(x), \quad a_{n,n-s} \neq 0,\, n \geq s,$$

which is a particular case of $(t+s, 2t)$-coherence.

On the other hand, in 2003 and 2004, M. Alfaro, F. Marcellán, A. Peña, and M. L. Rezola ([4] and [5]) analyzed the following algebraic relation between two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$

$$P_n(x) + a_n P_{n-1}(x) = Q_n(x) + b_n Q_{n-1}(x), \quad n \geq 1,$$

that we will call $(1,1)$-coherence of order 0. It yields the relation between the corresponding linear functionals

$$\left(x - C^P\right)\mathcal{U} = \eta\left(x - C^R\right)\mathcal{V}, \tag{0.0.9}$$

where C^P and C^R are constant numbers. They showed that, under some conditions, a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order 0 if and only if these functionals satisfy (0.0.9).

In this direction, in 2006, J. Petronilho ([105]) extended this problem when

$$P_n(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0,$$

where $a_{i,n} = 0 = b_{i,n}$ when $n - i < 0$, which is called $(M+1)$-$(N+1)$ *type linear structure relation* and, in our terminology, a (M, N)-coherence relation of order 0. He proved that if a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order 0, then under some conditions, there exist polynomials $\varphi(x)$ and $\rho(x)$ such that these functionals satisfy

$$\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}, \quad \text{with } \deg(\varphi(x)) = N, \deg(\rho(x)) = M. \tag{0.0.10}$$

Conversely, when $\mathcal{U}$ and $\mathcal{V}$ are related by (0.0.10), then

$$\sum_{i=n-M}^{n+N} a_{i,n,1} P_i(x) = \sum_{i=n-N}^{n+N} b_{i,n,1} Q_i(x), \quad n \geq 0,$$
$$\sum_{i=n-M}^{n+M} a_{i,n,2} P_i(x) = \sum_{i=n-N}^{n+M} b_{i,n,2} Q_i(x), \quad n \geq 0,$$

hold, where $\{a_{i,n,j}\}_{n\geq 0}$ and $\{b_{i,n,j}\}_{n\geq 0}$, $j = 1, 2$, are some sequences of complex numbers. Furthermore, in this work the case $(M, N) = (2, 1)$ is analyzed

In 2010, M. Alfaro, F. Marcellán, A. Peña, and M. L. Rezola ([6]) studied the $(M, 0)$-coherence relation of order 0 when the sequence of coefficients is constant. In particular, they analyzed the case $M = 2$, describing all the families of monic polynomials $\{P_n(x)\}_{n\geq 0}$ orthogonal with respect to a regular linear functional such that the new families $\{Q_n(x)\}_{n\geq 0}$ are also orthogonal. In 2011 ([7]), they studied $(2, 0)$-coherence of order 0, obtaining a characterization of the orthogonality of the sequence $\{Q_n(x)\}_{n\geq 0}$ in terms of the coefficients of a quadratic polynomial $\rho(x)$ such that $\mathcal{U} = \rho(x)\mathcal{V}$ holds. This kind of coherence also was analyzed by J. Wimp and H. Kiesel in [59] (1994) and [60] (1995), A. Branquinho and F. Marcellán in [24] (1996), and C. Hounga, M. N. Hounkonnou, and A. Ronveaux in [46] (2006).

In 2013, M. Alfaro, A. Peña, J. Petronilho, and M. L. Rezola ([9]) studied $(2, 1)$-coherent pairs of order 0 of regular linear functionals. They obtained characterizations for the orthogonality of the sequence $\{Q_n(x)\}_{n\geq 0}$ in terms of coherence coefficients, as well as of the coefficients of the polynomials $\varphi(x)$ and $\rho(x)$ which appear in the rational transformation (0.0.10) where $M = 2$ and $N = 1$.

Finally, in 2008, M. N. de Jesus and J. Petronilho ([52, 54]) proposed the more general case, which we will call (M, N)-*coherent pair of order* (m, k), where the derivatives of order m and k of two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are related by

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

M and N are non-negative integers, and, $\{a_{i,n}\}_{n\geq 0}$ and $\{b_{j,n}\}_{n\geq 0}$, with $0 \leq i \leq M$ and $0 \leq j \leq N$, are complex numbers satisfying some natural conditions. They proved that if $\mathcal{U}$ and $\mathcal{V}$ are the corresponding regular linear functionals associated with $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, then for $0 \leq m \leq k$, there exist four polynomials $\psi_{N+m+i}(x)$ and $\phi_{M+k+i}(x)$ of degree $N+m+i$ and $M+k+i$, respectively, $i = 0, 1$, such that

$$D^{k-m}\left(\psi_{N+m+i}(x)\mathcal{U}\right) = \phi_{M+k+i}(x)\mathcal{V}, \quad i = 0, 1,$$

where $D\mathcal{U}$ denotes the distributional derivative of the linear functional $\mathcal{U}$. Therefore, if $k = m$ then the linear functionals $\mathcal{U}$ and $\mathcal{V}$ are related by a relation of rational type. Besides, they concluded that if $k = m+1$ and $\{P_n(x)\}_{n\geq 0} \neq \{Q_n(x)\}_{n\geq 0}$, then there exist polynomials $\varphi(x)$, $\rho(x)$, and $\tau(x)$ of degrees at most $2(N+m)$, $M+N+2m+2$, and $M+N+2m+1$, respectively, such that

$$\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}, \quad D(\rho(x)\mathcal{U}) = \tau(x)\mathcal{U},$$
$$D(\varphi(x)\rho(x)\mathcal{V}) = [2\varphi'(x)\rho(x) + \varphi(x)(\tau(x) - \rho'(x))]\mathcal{V},$$

and, hence, $\mathcal{U}$ and $\mathcal{V}$ are semiclassical linear functionals of classes at most $M+N+2m$ and $M+3N+4m$, respectively. When $k = m+1$ and $\{P_n(x)\}_{n\geq 0} = \{Q_n(x)\}_{n\geq 0}$, $\mathcal{U}$ and $\mathcal{V}$ coincide up to a constant factor and they are semiclassical linear functionals of class at most $\max\{M+m, N+m-2\}$.

Afterwards, in 2013, in [52, 55], M. N. de Jesus and J. Petronilho showed that if the regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ are related by an expression of rational type (0.0.10) and $\mathcal{V}$ is a semiclassical linear functional given by $D(\sigma(x)\mathcal{V}) = \tau(x)\mathcal{V}$ with $\deg(\sigma(x)) = \ell$ and $\deg(\tau(x)) = t \geq 1$, then $(\mathcal{U}, \mathcal{V})$ is a $(M+2\ell+N, s+\ell+2N)$-coherent pair, with $s = \max\{\ell-2, t-1\}$. Also, they concluded that the (M, N)-coherence relation for a pair of positive Borel measures on the real line (μ_0, μ_1), with finite moments of all orders, given by

$$\frac{P'_{n+1}(x)}{n+1} + \sum_{i=1}^{M} a_{i,n}\frac{P'_{n-i+1}(x)}{n-i+1} = Q_n(x) + \sum_{i=1}^{N} b_{i,n}Q_{n-i}(x), \quad n \geq 0,$$

is a sufficient condition for the following algebraic relation

$$P_{n+1}(x) + \sum_{i=1}^{M}\frac{n+1}{n-i+1}a_{i,n}P_{n-i+1}(x) = S_{n+1}(x;\lambda) + \sum_{j=1}^{\max\{M,N\}} c_{j,n,\lambda}S_{n-j+1}(x;\lambda), \quad n \geq 0,$$

where $\{S_n(x;\lambda)\}_{n\geq 0}$ is the Sobolev SMOP associated with the Sobolev inner product given by (0.0.3). In this way, they extended the algorithms given by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna in [48], and by K. H. Kwon, J. H. Lee and F. Marcellán in [66], in the cases when (μ_0, μ_1) is a $(1,0)$-coherent pair and a $(2,0)$-coherent pair, respectively, to compute the Sobolev-Fourier coefficients $\{f_n(\lambda)/s_n(\lambda)\}_{n\geq 0}$ (see (0.0.4)-(0.0.6)) when (μ_0, μ_1) is a (M, N)-coherent pair without the need to form the Sobolev orthogonal polynomials $\{S_n(x;\lambda)\}_{n\geq 0}$ explicitly.

To complete this historical summary about coherent pairs on the real line, in 2012, A. Branquinho and M. N. Rebocho ([25]), proved that if a pair of positive Borel measures (μ_0, μ_1), with finite moments of all orders, is a $(1,0)$-coherent pair of order 2, then each of them is semiclassical and μ_1 is a rational modification of μ_0. Also, they stated that, under some conditions, $\mathcal{U}$ is a semiclassical linear functional of class at most N if and only if $(\mathcal{U}, \mathcal{U})$ is a $(N+2, N)$-coherent pair.

On the other hand, W. Hahn ([44], 1949) noted that the characterizations of the classical SMOP based on the derivative operator (the classical orthogonal polynomials are only those whose sequence of derivatives is also orthogonal) and eigenfunctions of second order linear differential operators are too restrictive, because excluding orthogonal polynomials of a discrete variable. So, he introduced a more general linear operator: the $\mathbf{L}_{(q,\omega)}$-*derivative or Hahn operator*, with parameters q and ω, given by

$$\left(\mathbf{L}_{(q,\omega)} f\right)(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x + \omega}.$$

This operator includes, as particular cases, the *difference operator with step* ω and the q-*derivative* operator

$$D_\omega = \mathbf{L}_{(1,\omega)} \text{ with } \omega \neq 0, \quad \text{and} \quad D_q = \mathbf{L}_{(q,0)} \text{ with } q \neq 1,$$

respectively. Notice that we get the standard derivative when $\omega \to 0$ and $q \to 1$.

In this way, the notion of coherent pair was extended to the theory of orthogonal polynomials of a discrete variable by I. Area, E. Godoy, and F. Marcellán ([14, 15, 16, 17]).

2. Coherent Pairs and Uniform Lattices

In 2000 ([14, 15, 17]), they generalized the notion of coherent pair to D_ω-coherent pair using the difference operator D_ω

$$\left(D_\omega p\right)(x) = \frac{p(x+\omega) - p(x)}{\omega}, \quad \omega \in \mathbb{C} \setminus \{0\},$$

where $p(x)$ is a polynomial with complex coefficients, as follows. A pair of weakly quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$, of order $M_0 \geq 2$ and $M_1 \geq 1$, respectively, is called a D_ω-*coherent pair* (in our terminology, $(1,0)$-D_ω-*coherent pair*) if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$\frac{D_\omega P_{n+1}(x)}{n+1} + a_n \frac{D_\omega P_n(x)}{n} = Q_n(x), \quad a_n \neq 0, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\}.$$

They proved that if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-D_ω-coherent pair, then at least one of the weakly quasi-definite linear functionals must be D_ω-classical, as well as, they are related by an expression of rational type. More precisely, assuming $M_1 > 7$, there are two possibilities:

- $\mathcal{U}$ is a D_ω-classical linear functional and there exist polynomials $\sigma_{\mathcal{U}}(x)$, $\tau_{\mathcal{U}}(x)$, and $\rho_{\mathcal{U}}(x)$, with $\deg(\sigma_{\mathcal{U}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{U}}(x)) = \deg(\rho_{\mathcal{U}}(x)) = 1$, such that

$$D_\omega(\sigma_{\mathcal{U}}(x-\omega)\mathcal{U}) = \tau_{\mathcal{U}}(x)\mathcal{U} \quad \text{and} \quad \sigma_{\mathcal{U}}(x-\omega)\mathcal{U} = \rho_{\mathcal{U}}(x)\mathcal{V}.$$

- $\mathcal{V}$ is a D_ω-classical linear functional and there exist polynomials $\sigma_{\mathcal{V}}(x)$, $\tau_{\mathcal{V}}(x)$, and $\rho_{\mathcal{V}}(x)$, with $\deg(\sigma_{\mathcal{V}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{V}}(x)) = \deg(\rho_{\mathcal{V}}(x)) = 1$, such that

$$D_\omega(\sigma_{\mathcal{V}}(x+\omega)\mathcal{V}) = \tau_{\mathcal{V}}(x)\mathcal{V} \quad \text{and} \quad \sigma_{\mathcal{V}}(x)\mathcal{U} = \rho_{\mathcal{V}}(x)\mathcal{V}.$$

Also, they determined all $(1,0)$-D_1-coherent pairs for classical discrete SMOP (Charlier, Meixner, Kravchuk, Hahn) and their companions. Finally, using a limit process when $\omega \to 0$ (from Hahn to Jacobi, Meixner to Laguerre, Charlier to Hermite), they recovered the classification of $(1,0)$-coherent pairs of positive definite linear functionals given by Meijer in [99].

Later on, in 2004, K. H. Kwon, J. H. Lee, and F. Marcellán ([67]) generalized the concept of D_ω-coherent pair to *($M+1$-term) generalized D_ω-coherent pair*, taking into account the inner product

$$\langle p(x), r(x)\rangle_{\lambda,\omega} = \int_{\mathbb{R}} p(x)r(x)d\mu_0(x) + \lambda\sum_{j=1}^{\infty} D_\omega p(x_j)D_\omega r(x_j)d\mu_1(x_j), \quad \lambda > 0, \quad (0.0.11)$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients, μ_0 is a nontrivial probability measure and μ_1 is a discrete measure supported on a uniform lattice $\{x_j\}_{j\geq 0}$ with step ω (Notice that when $\omega \to 0$, (0.0.11) becomes a Sobolev inner product in the standard sense like (0.0.3)). The pair of measures (μ_0, μ_1) is said to be a ($M+1$-term) generalized D_ω-coherent pair, for us $(M,0)$-D_ω-coherent pair, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$\frac{D_\omega P_{n+1}(x)}{n+1} + \sum_{j=1}^{M} a_{j,n}\frac{D_\omega P_{n-j+1}(x)}{n-j+1} = Q_n(x), \quad a_{M,n} \neq 0,\ n \geq M.$$

They showed that this $(M,0)$-D_ω-coherence condition yields the relation

$$P_{n+1}(x) + \sum_{j=1}^{M} \frac{n+1}{n-j+1}a_{j,n}P_{n-j+1}(x)$$
$$= S_{n+1}(x;\lambda,\omega) + \sum_{j=1}^{M} c_{j,n,\lambda,\omega}S_{n-j+1}(x;\lambda,\omega), \quad n \geq M, \quad (0.0.12)$$

where $\{c_{n,\lambda,\omega}\}_{n\geq M}$ are rational functions in $\lambda > 0$, $c_{M,n,\lambda,\omega} \neq 0$, $a_{M,n} \neq 0$, and $\{S_n(x;\lambda,\omega)\}_{n\geq 0}$ is the SMOP associated with the inner product (0.0.11). Conversely, if (0.0.12) holds, then (μ_0, μ_1) is a (M, M)-D_ω-coherent pair of measures.

Additionally, they analyzed the case $\omega = 1$ and $M = 2$ for regular linear functionals $\mathcal{U}, \mathcal{V}$. They concluded that if $(\mathcal{U}, \mathcal{V})$ is a $(2, 0)$-D_1-coherent pair, then the linear functionals are D_1-semiclassical (of class ≤ 6 for $\mathcal{U}$ and of class ≤ 2 for $\mathcal{V}$), and they are related by an expression of rational type. Also, they studied the cases when either $\mathcal{U}$ or $\mathcal{V}$ is a D_1-classical linear functional.

3. Coherent Pairs and q-Lattices

In 2002, other extension of the concept of coherent pair to the orthogonal polynomials theory of a discrete variable was done by I. Area, E. Godoy, and F. Marcellán ([14, 16]), using the q-derivative operator D_q

$$(D_q p)(x) = \frac{p(qx) - p(x)}{(q-1)x},\ x \neq 0, \quad \text{and} \quad (D_q p)(0) = p'(0), \quad q \in \mathbb{C} \setminus \{0, 1\},$$

where $p(x)$ is a polynomial with complex coefficients. In this case, a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a *q-coherent pair* (in our terminology, a *$(1, 0)$-q-coherent pair*) if

$$\frac{(D_q P_{n+1})(x)}{[n+1]_q} + a_n \frac{(D_q P_n)(x)}{[n]_q} = Q_n(x), \qquad a_n \neq 0,\ n \geq 1,$$
$$0 < q < 1 \ \text{ and } \ [n]_q = \frac{q^n - 1}{q - 1}, \qquad n \geq 1,$$

holds, where $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the SMOP with respect to $\mathcal{U}$ and $\mathcal{V}$, respectively. They deduced that if $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-q-coherent pair of linear functionals, then at least one of them must be q-classical and one is a rational modification of the other. More precisely, two cases appear:

- $\mathcal{U}$ is a D_q-classical linear functional and there exist polynomials $\sigma_{\mathcal{U}}(x)$, $\tau_{\mathcal{U}}(x)$, and $\rho_{\mathcal{U}}(x)$, with $\deg(\sigma_{\mathcal{U}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{U}}(x)) = \deg(\rho_{\mathcal{U}}(x)) = 1$, such that

$$D_q\left[\sigma_{\mathcal{U}}(q^{-1}x)\mathcal{U}\right] = \tau_{\mathcal{U}}(x)\mathcal{U} \quad \text{and} \quad \sigma_{\mathcal{U}}(q^{-1}x)\mathcal{U} = \rho_{\mathcal{U}}(x)\mathcal{V}.$$

- $\mathcal{V}$ is a D_q-classical linear functional and there exist polynomials $\sigma_{\mathcal{V}}(x)$, $\tau_{\mathcal{V}}(x)$, and $\rho_{\mathcal{V}}(x)$, with $\deg(\sigma_{\mathcal{V}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{V}}(x)) = \deg(\rho_{\mathcal{V}}(x)) = 1$, such that

$$D_q\left[\sigma_{\mathcal{V}}(qx)\mathcal{V}\right] = \tau_{\mathcal{V}}(x)\mathcal{V} \quad \text{and} \quad \sigma_{\mathcal{V}}(x)\mathcal{U} = \rho_{\mathcal{V}}(x)\mathcal{V}.$$

Besides, they determined all companions of $\mathcal{U}$ or $\mathcal{V}$ when one of them is either Big q-Jacobi, or Little q-Jacobi or Little q-Laguerre/Wall and $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-q-coherent pairs of positive-definite linear functionals. Moreover, by using a limit process when $q \to 1$ in the Little q-Jacobi and Little q-Laguerre/Wall cases, the classification of all $(1,0)$-coherent pairs of linear functionals given by Meijer in [99] for the continuous case was reached. Additionally, substituting the variable x by q^{-x} and using limit processes when $q \to 1$ again (from q-Hahn to Hahn, q-Meixner to Meixner, Quantum q-Kravchuk to Kravchuk, and Alternative q-Charlier to Charlier), they recovered the classification of the $(1,0)$-D_1-coherent pairs of nonnegative-definite linear functionals given in [15].

4. Coherent Pairs of Measures on the Unit Circle

Finally, in 2008, A. Branquinho, A. Foulquié Moreno, F. Marcellán, and M. N. Rebocho ([23]) extended the notion of coherent pair to positive definite Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$, defined on the linear space of Laurent polynomials and associated with positive Borel measures μ_0 and μ_1, respectively, with infinite support on the unit circle $\mathbb{T}$ and with finite moments of all orders. They considered the Sobolev inner product

$$\langle p(z), r(z)\rangle_\lambda = \int_{\mathbb{T}} p(z)\overline{r(z)}d\mu_0(z) + \lambda \int_{\mathbb{T}} p'(z)\overline{r'(z)}d\mu_1(z), \quad \lambda > 0,$$

where $p(z)$ and $r(z)$ are polynomials with complex coefficients, and $\{S_n(z;\lambda)\}_{n\geq 0}$ is its corresponding SMOP. If the sequences of monic orthogonal polynomials on the unit circle (OPUC) $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ with respect to μ_0 and μ_1, respectively, satisfy

$$\frac{\phi'_{n+1}(z)}{n+1} + a_n \frac{\phi'_n(z)}{n} = \psi_n(z), \quad a_n \neq 0,\ n \geq 1,$$

then, (μ_0, μ_1) is said to be a *coherent pair* (in our terminology, a $(1,0)$-*coherent pair*) on the unit circle. They proved that this coherence relation is a sufficient condition for the algebraic relation

$$\phi_{n+1}(z) + \frac{n+1}{n} a_n \phi_n(z) = S_{n+1}(z;\lambda) + c_{n,\lambda} S_n(z;\lambda), \quad n \geq 1,$$

where the sequence $\{c_{n,\lambda}\}_{n\geq 1}$ is given by

$$c_{n,\lambda} = \frac{\varpi_{n-1}(\lambda)}{\varpi_n(\lambda)}, \quad n \geq 1,$$

being $\{\varpi_n(\lambda)\}_{n\geq 0}$ a sequence of orthogonal polynomials associated with a positive Borel measure supported on the real line with finite moments of all orders.

Besides, they concluded that if $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-coherent pair of regular Hermitian linear functionals, then $\mathcal{U}$ is a semiclassical linear functional, $\mathcal{V}$ is a rational transformation of $\mathcal{U}$, and $\{\psi_n(z)\}_{n\geq 0}$ is a quasi-orthogonal sequence of order at most 6 with respect to the functional $[zA(z)+(1/z)\overline{A}(1/z)]\mathcal{U}$, with $A(z)$ a polynomial with complex coefficients. Additionally, they analyzed the cases when either μ_0 or μ_1 is the Lebesgue measure or μ_0 is the Bernstein-Szegő measure, showing that if (μ_0, μ_1) is a $(1,0)$-coherent pair and

- if μ_0 is the Lebesgue measure, then μ_1 is the Bernstein-Szegő measure

$$d\mu_1 = \frac{1}{|z+C|^2}\frac{d\theta}{2\pi}, \quad |C|<1,\ z=e^{i\theta};$$

- if μ_1 is the Lebesgue measure, then μ_0 must be the absolutely continuous measure

$$d\mu_0 = |z-\kappa|^2\frac{d\theta}{2\pi}, \quad z=e^{i\theta}.$$

Afterwards, in 2010, A. Branquinho and M. N. Rebocho ([26]) studied the following more general relation

$$\sum_{j=0}^{M_1} a_{j,n,1}\frac{\phi'_{n+M_1-j+1}(z)}{n+M_1-j+1} + \sum_{j=0}^{M_2} a_{j,n,2}\left(\phi^*_{n+M_2-j}(z)\right)' \\ = \sum_{j=0}^{N_1} b_{j,n,1}\psi_{n+N_1-j}(z) + \sum_{j=0}^{N_2} b_{j,n,2}\psi^*_{n+N_2-j}(z), \quad n\geq 0 \quad (0.0.13)$$

with $M_1 = N_1$, $\max\{M_2, N_2\} < M_1$ and some extra conditions. In this case, they showed that $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ must be semiclassical sequences of monic OPUC. Moreover, when $\phi_n(z) = \psi_n(z)$ for all n, and, under some extra conditions and special values of M_i, N_i, $i=1,2$, (0.0.13) is a sufficient and necessary condition for the semiclassical character of $\{\phi_n(z)\}_{n\geq 0}$. Finally, they analyzed the $(0,1)$-coherence case

$$\frac{\phi'_{n+1}(z)}{n+1} = \psi_n(z) + b_n\psi_{n-1}(z), \quad b_n\neq 0,\ n\geq 1,$$

when $\mathcal{U}$ is the positive definite linear functional associated with either the Lebesgue measure or the Bernstein-Szegő measure, concluding that if $(\mathcal{U}, \mathcal{V})$ is a $(0,1)$-coherent pair of regular Hermitian linear functionals on the unit circle and

- if $\mathcal{U}$ is the Lebesgue linear functional, then the linear functional $\mathcal{V}$ is a transformation of $\mathcal{U}$, given as
$$\mathcal{V} = \left(1+\bar{b}_1 z + b_1 z^{-1}\right)\mathcal{U};$$
- if $\mathcal{U}$ is the Bernstein-Szegő linear functional, then $\mathcal{V}$ is the Lebesgue linear functional.

Original Contributions of this Dissertation

In Chapter 2, we analyze (M,N)-coherent pairs of order (m,k) of sequences of orthogonal polynomials of a continuous variable on the real line. In Chapter 3, we study (M,N)-coherent pairs of order (m,k) in the theory of orthogonal polynomials of a discrete variable, the uniform lattice and the q-lattice, respectively. Finally, Chapter 4 is focused on the study of (M,N)-coherent pairs on the unit circle. Next, we present the original results obtained in each of these chapters.

Chapter 2: Coherent Pairs, Continuous Case.

In this chapter, we study (M,N)-coherent pairs of order (m,k) of regular linear functionals in the theory of orthogonal polynomials of one (continuous) variable. Two regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ constitute a (M,N)-*coherent pair of order* (m,k), if their respective SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, satisfy an algebraic relation

$$P_n^{[m]}(x)+\sum_{i=1}^{M} a_{i,n}P_{n-i}^{[m]}(x)=Q_n^{[k]}(x)+\sum_{i=1}^{N} b_{i,n}Q_{n-i}^{[k]}(x), \quad n\geq 0,$$

where M,N,m,k are fixed non-negative integers, $\{a_{i,n}\}_{n\geq 0},\{b_{i,n}\}_{n\geq 0}\subset\mathbb{C}$, $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, $a_{i,n}=b_{i,n}=0$ when $i>n$, and $P_n^{[m]}(x)$, $n\geq 0$, are the monic polynomials

$$P_n^{[m]}(x)=\frac{P_{n+m}^{(m)}(x)}{(n+1)_m}, \quad m,n\geq 0.$$

When $k=0$, $(\mathcal{U},\mathcal{V})$ is called a (M,N)-*coherent pair of order* m, and it is said to be a (M,N)-*coherent pair* if also $m=1$.

In Section 2.1, we prove that if a pair of linear functionals $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order (m,k), then they must be semiclassical, whenever $m\neq k$. Furthermore, assuming $m\geq k$, there exist polynomials $\varphi(x)$, $\rho(x)$, $\phi_{M+k+n}(x)$, and $\psi_{N+m+n}(x)$ with $\deg(\phi_{M+k+n}(x))=M+k+n$ and $\deg(\psi_{N+m+n}(x))=N+m+n$, such that these linear functionals are related by

$$\begin{gathered}D^{m-k}[\phi_{M+k+n}(x)\mathcal{V}]=\psi_{N+m+n}(x)\mathcal{U}, \quad n\geq 0,\\ \varphi(x)\mathcal{U}=\rho(x)\mathcal{V}.\end{gathered}$$

Additionally, their formal Stieltjes series, $S_{\mathcal{V}}$ and $S_{\mathcal{U}}$, satisfy

$$\sum_{i=0}^{m-k} B_{i,n}(z)S_{\mathcal{V}}^{(i)}(z)=C_n(z), \quad n\geq 0,$$

$$\psi_{N+m+n}(z)S_{\mathcal{U}}(z) - [\phi_{M+k+n}(z)S_{\mathcal{V}}(z)]^{(m-k)} = A_n(z), \quad n \geq 0,$$

where $B_{i,n}(z)$, $C_n(z)$, and $A_n(z)$ are polynomials of degree at most $M+N+2k+2n+i+1$, $2n+N+\max\{M+2k, N+2m\}$, and $n-1+\max\{M+2k-m, N+m\}$, respectively. Therefore, we can obtain $S_{\mathcal{V}}$, and hence $S_{\mathcal{U}}$, by solving (formally) these differential equations for some $n \geq 0$.

This generalizes some previous results given by F. Marcellán, T. Pérez, and M. Piñar in [73] and H. G. Meijer in [99] for $(1,0)$-coherent pairs, A. Delgado and F. Marcellán in [32, 33] for $(1,1)$-coherent pairs, K. H. Kwon, J. H. Lee, and F. Marcellán in [66] for $(2,0)$-coherent pairs, P. Maroni and R. Sfaxi in [88, 90] for $(M+N, 2M+1)$-coherent pairs, M. Alfaro, F. Marcellán, A. Peña, and M. L. Rezola in [4, 5] for $(1,1)$-coherent pairs of order 0, J. Petronilho in [105] for (M,N)-coherent pairs of order 0, M. N. de Jesus and J. Petronilho in [52, 54] for (M,N)-coherent pairs of order $(k+1,k)$ and (k,k), A. Branquinho and M. N. Rebocho in [25] for $(1,0)$-coherent pairs of order 2, and, F. Marcellán and N. C. Pinzón-Cortés in [79] for $(1,1)$-coherent pairs of order m.

In Section 2.2, we consider the relationship between (M,N)-coherent pairs of order m, $m \geq 1$, of SMOP associated with positive linear functionals and the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$ with respect to the inner product

$$\langle p(x), r(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)r(x)d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x)r^{(m)}(x)d\mu_1, \quad \lambda > 0, \tag{0.0.14}$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients, and μ_0 and μ_1 are the measures associated with the linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively, such that they constitute a (M,N)-coherent pair of order m.

We extend a special property stated by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna in [48] for $(1,0)$-coherent pairs, M. G. de Bruin and H. G. Meijer in [27] for $(2,0)$-coherent pairs, A. Delgado and F. Marcellán in [32, 33] for $(1,1)$-coherent pairs, F. Marcellán, A. Martínez-Finkelshtein, and J. Moreno-Balcázar in [71] for $(M+1,0)$-coherent pairs, and, M. N. de Jesus and J. Petronilho in [52, 55] for (M,N)-coherent pairs, showing that if (μ_0,μ_1) is a (M,N)-coherent pair of order m, then

$$P_{n+m}(x) + \sum_{i=1}^{M} \frac{(n+1)_m a_{i,n}}{(n-i+1)_m} P_{n-i+m}(x) = S_{n+m}(x;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda} S_{n-j+m}(x;\lambda), \quad n \geq 0, \tag{0.0.15}$$

holds, where $K = \max\{M,N\}$ and $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, are rational functions in λ for which we present explicit and recursive expressions. Conversely, if (0.0.15) holds, then (μ_0,μ_1) is a (M,K)-coherent pair of order m.

Furthermore, when (μ_0,μ_1) is a (M,N)-coherent pair of order m, we obtain an algorithm to compute the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$ in (0.0.15), as well as the

Fourier-Sobolev coefficients $\{f_n/s_n\}_{n\geq 0}$ such that

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} S_n(x;\lambda),$$

where

$$f_n = \left\langle f(x), S_n(x;\lambda)\right\rangle_\lambda \quad \text{and} \quad s_n = \left\langle S_n(x;\lambda), S_n(x;\lambda)\right\rangle_\lambda, \quad n \geq 0,$$

for a function $f(x)$ in the weighted Sobolev linear space

$$W^{m,2}[I,\mu_0,\mu_1] = \left\{ g : I \to \mathbb{R} \mid g \in L^2_{\mu_0}(I)\,,\, g^{(m)} \in L^2_{\mu_1}(I) \right\}.$$

For this end, we get the relations

$$s_{n+m} c_{j,n+j,\lambda} = \zeta_{K-j,n+j,\lambda} - \sum_{i=1}^{K-j} c_{i,n,\lambda} c_{j+i,n+j,\lambda} s_{n+m-i}, \quad 0 \leq j \leq K, n \geq 0,$$

$$f_{n+m} + \sum_{j=1}^{K} c_{j,n,\lambda} f_{n-j+m} = \varrho_{n,\lambda,f}, \quad n \geq 0,$$

with initial conditions

$$c_{j,n,\lambda} = 0,\ j > K \text{ or } n < j \leq K, \quad c_{0,n,\lambda} = 1, \quad n \geq 0,$$
$$s_n = \left\langle P_n(x), P_n(x)\right\rangle_{\mu_0}, \quad f_n = \left\langle f(x), P_n(x)\right\rangle_{\mu_0}, \quad 0 \leq n < m,$$

where the constants $\zeta_{K-j,n+j,\lambda}$, $0 \leq j \leq K$, and $\varrho_{n,\lambda,f}$, for $n \geq 0$, depend on λ, the coherence coefficients and the SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$. Notice that we do not need to know the explicit expressions of the Sobolev orthogonal polynomials $S_n(x;\lambda)$, $n \geq 0$, for obtaining Fourier-Sobolev coefficients, although, using $S_n(x;\lambda) = P_n(x)$ for $n < m$, and (0.0.15) we can compute them. In addition, a numerical example illustrating the algorithm is given.

This algorithm is a generalization of those obtained in [48], [66] and [52, 55] for $(1,0)$, $(2,0)$ and (M,N) -coherent pairs, respectively.

Finally, we prove additional properties satisfied by the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, and the Sobolev norms $\{s_n\}_{n\geq 0}$ in both $(1,0)$ and $(1,1)$ -coherence cases of order m.

To complete this chapter, in Section 2.3, we analyze (M,N)-coherent pairs of regular linear functionals from a matrix perspective by considering a pair of matrices $\mathcal{M}_P$, $\mathcal{M}_Q$ which are similar to the monic Jacobi matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. In this way, if $[\mathcal{M}_P, \mathcal{M}_Q]$ denotes the commutator of the matrices $\mathcal{M}_P$ and $\mathcal{M}_Q$

$$[\mathcal{M}_P, \mathcal{M}_Q] = \mathcal{M}_P \mathcal{M}_Q - \mathcal{M}_Q \mathcal{M}_P, \tag{0.0.16}$$

we show that

$$(\mathcal{M}_Q - \mathcal{M}_P)^2 = [\mathcal{M}_P, \mathcal{M}_Q]$$

is a necessary condition for (M, N)-coherence condition. Besides, when $\mathcal{U}$ is a classical linear functional, we can regard $\mathcal{M}_{P[m]}$, a similar matrix to the monic Jacobi matrix associated with the SMOP $\{P_n^{[m]}(x)\}_{n\geq 0}$, and in this case we get

$$\mathcal{M}_{P[m]} = \mathcal{M}_Q$$

is a necessary condition for $(\mathcal{U}, \mathcal{V})$ to be a (M, N)-coherent pair of order m, $m \geq 0$.

Furthermore, when $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order m, $m \geq 1$, of positive definite linear functionals, we recall the monic Sobolev polynomials $S_n(x; \lambda)$, $n \geq 0$, orthogonal with respect to the inner product (0.0.14), and we prove

$$x\mathbf{s}(x; \lambda) = \mathcal{H}_{S,P,\lambda}\mathbf{s}(x; \lambda),$$

where $\mathbf{s}(x; \lambda) = \begin{bmatrix} S_0(x; \lambda), & S_1(x; \lambda), & \cdots \end{bmatrix}^T$ and $\mathcal{H}_{S,P,\lambda}$ is a Hessenberg matrix similar to the monic Jacobi matrix associated with the linear functional $\mathcal{U}$.

The results of this chapter yield the following publications

[53] M. N. de Jesus, F. Marcellán, J. Petronilho, and N. C. Pinzón-Cortés. (M, N)-*Coherent Pairs of Order* (m, k) *and Sobolev Orthogonal Polynomials*. J. Comput. Appl. Math. **256**, 16-35, (2014).

[79] F. Marcellán and N. C. Pinzón-Cortés. *Higher Order Coherent Pairs*. Acta Appl. Math. **121** (1), 105-135, (2012).

[80] F. Marcellán and N. C. Pinzón-Cortés. (M, N)-*Coherent Pairs of Linear Functionals and Jacobi Matrices*. Submitted.

Chapter 3: D_ν-Coherent Pairs, for $\nu = \omega, q$.

In this chapter,

ν and ν^* will denote either $\omega \in \mathbb{C} \setminus \{0\}$ and $-\omega$, or $q \in \mathbb{C} \setminus \{0, 1\}$ and q^{-1}.

We introduce the notion of (M, N)-*D_ν-coherent pair of order* (m, k) of regular linear functionals $(\mathcal{U}, \mathcal{V})$, for fixed $M, N, m, k \in \mathbb{N} \cup \{0\}$, as follows. Their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n^{[k,\nu]}(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}^{[k,\nu]}(x), \quad n \geq 0,$$

where $a_{i,n}, b_{i,n} \in \mathbb{C}$ for $n \geq 0$, $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, $a_{i,n} = b_{i,n} = 0$ for $i > n$, and

$$P_n^{[m,\nu]}(x) = \frac{D_\nu^m P_{n+m}(x)}{\eta_{n,m,\nu}}, \quad n \geq 0,$$

where the constants $\eta_{n,m,\nu}$ are such that $P_n^{[m,\nu]}(x)$ is a monic polynomial, and the derivatives D_ω and D_q are defined by

$$\left(D_\omega p\right)(x) = \frac{p(x+\omega) - p(x)}{\omega}, \qquad \left(D_q p\right)(x) = \frac{p(qx) - p(x)}{(q-1)x},$$

for every polynomial $p(x)$ with complex coefficients. If $k = 0$, $(\mathcal{U}, \mathcal{V})$ is said to be a (M,N)-D_ν-*coherent pair of order* m, and in this case, when $m = 1$, we will say that $(\mathcal{U}, \mathcal{V})$ is a (M,N)-D_ν-*coherent pair*. On the other hand, when $\mathcal{U}$ and $\mathcal{V}$ are weakly quasi-definite linear functionals of order Υ_0 and Υ_1, respectively, and $\{P_n(x)\}_{n=0}^{\Upsilon_0}$ and $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$ are their corresponding families of monic orthogonal polynomials, all theses definitions hold with the additional restrictions $0 \leq n \leq \min\{\Upsilon_0 - m, \Upsilon_1 - k\}$, $0 \leq M, m \leq \Upsilon_0$, and $0 \leq N, k \leq \Upsilon_1$.

In Section 3.1, we study $(1,1)$-D_ν-coherent pairs of regular functionals and we prove that these linear functionals must be D_ν-semiclassical, one of them of class at most 1 and the another of class at most 5, and they are related by an expression of rational type as

$$\tilde{\sigma}(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V}, \quad \deg\left(\tilde{\sigma}(x;\nu)\right) \leq 3, \deg\left(\rho(x;\nu)\right) = 1.$$

This generalizes the results obtained by I. Area, E. Godoy, and F. Marcellán in [14, 15, 16, 17] for $(1,0)$-D_ν-coherent pairs and we get the D_ν-analogue results obtained by A. Delgado and F. Marcellán in [33] for $(1,1)$-coherent pairs. Besides, we analyze the case of $(1,0)$-D_ν-coherent pairs and we recover the results obtained by I. Area, E. Godoy, and F. Marcellán.

Additionally, we study the case when $\mathcal{U}$ is a D_ν-classical linear functional. For the $(1,0)$-D_ν-coherence case, we conclude that $\mathcal{V}$ is a D_ν-semiclassical linear functional of class at most 1, and for the $(1,1)$-D_ν-coherence case, it is D_ν-semiclassical of class at most 2. In both cases, we give the explicit rational expression which they satisfy.

Furthermore, the case when $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$ and $(1,0)$ -D_ν-coherent pair of weakly quasi-definite linear functionals is described.

In Section 3.2, we survey the general case of (M,N)-D_ν-coherence of order (m,k). We show that if $(\mathcal{U}, \mathcal{V})$ is a (M,N)-D_ν-coherent pair of order (m,k) of regular linear functionals with $m \geq k$, then

$$D_{\nu^*}^{m-k}[\phi_{M+k+n}(x;\nu)\mathcal{V}] = \psi_{N+m+n}(x;\nu)\mathcal{U}, \quad n \geq 0,$$
$$\varphi(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V},$$

where $\phi_{M+k+n}(x;\nu)$, $\psi_{N+m+n}(x;\nu)$, $\varphi(x;\nu)$ and $\rho(x;\nu)$ are polynomials such that deg $(\phi_{M+k+n}(x;\nu)) = M+k+n$ and $\deg(\psi_{N+m+n}(x;\nu)) = N+m+n$, $n \geq 0$, i.e., the linear functionals $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor. Moreover, these relations allow us to infer that $\mathcal{U}$ and $\mathcal{V}$ are D_ν-semiclassical linear functionals when $m \neq k$.

Conversely, if $\mathcal{U}$ and $\mathcal{V}$ are two D_ν-semiclassical linear functionals related by a expression of rational type, i.e., there exist monic polynomials $\sigma(x)$ and $\varphi(x)$, as well as nonzero polynomials $\tau(x)$ and $\rho(x)$, such that

$$D_{\nu^*}\left[\sigma(x)\mathcal{V}\right] = \tau(x)\mathcal{V}, \quad \text{and} \quad \varphi(x)\mathcal{U} = \rho(x)\mathcal{V},$$
$$\deg(\sigma(x)) = \ell, \quad \deg(\tau(x)) = t \geq 1, \quad \deg(\varphi(x)) = \jmath, \quad \deg(\rho(x)) = r,$$

then

$$\sum_{i=n-r-\ell}^{n+\jmath+\ell} a_{i,n} P_i^{[1,\nu]}(x) = \sum_{i=n-\jmath-s}^{n+\jmath+\ell} b_{i,n} Q_i(x),$$

where $a_{n+\jmath+\ell,n} b_{n+\jmath+\ell,n} \neq 0$, for $n \geq 0$, and $s = \max\{\ell-2, t-1\}$. Thus, $(\mathcal{U},\mathcal{V})$ is a $(\jmath+2\ell+r, 2\jmath+\ell+s)$-$D_\nu$-coherent pair.

In Section 3.3, we focus our attention on the Sobolev inner product

$$\langle p(x), r(x)\rangle_{\lambda,\nu} = \langle \mathcal{U}, p(x)r(x)\rangle + \lambda \langle \mathcal{V}, (D_\nu^m p)(x)(D_\nu^m r)(x)\rangle, \quad \lambda > 0, \tag{0.0.17}$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients, and $\mathcal{U}$ and $\mathcal{V}$ constitute a (M,N)-D_ν-coherent pair of order m. In this way, we prove that if the (M,N)-D_ν-coherence relation of order m holds, then

$$P_{n+m}(x) + \sum_{i=1}^{M} \frac{\eta_{n,m,\nu} a_{i,n}}{\eta_{n-i,m,\nu}} P_{n-i+m}(x) = S_{n+m}(x;\lambda,\nu) + \sum_{j=1}^{K} c_{j,n,\lambda,\nu} S_{n-j+m}(x;\lambda,\nu), \quad n \geq 0, \tag{0.0.18}$$

where $K = \max\{M,N\}$, $c_{j,n,\lambda,\nu}$ for $1 \leq j \leq K$, $n \geq 0$, are rational functions in the variable λ such that $c_{j,n,\lambda,\nu} = 0$, $n < j \leq K$, and $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ is the SMOP with respect to (0.0.17), (this is a generalization of the results obtained by K. H. Kwon, J. H. Lee, and F. Marcellán for $(M,0)$-D_ω-coherence in [67]). Conversely, (M,K)-D_ν-coherence of order m is a necessary condition for (0.0.18).

Besides, we state expressions which allows us to compute the coefficients $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1 \leq j \leq K$, and, consequently, we get the Sobolev orthogonal polynomials $S_n(x;\lambda,\nu)$, $n \geq 0$, since $S_n(x;\lambda,\nu) = P_n(x)$ for $n < m$.

On the other hand, when $(M,N) = (1,0)$ and $(M,N) = (1,1)$, we give some additional properties that the sequences $\{\langle S_n(x;\lambda,\nu), S_n(x;\lambda,\nu)\rangle_{\lambda,\nu}\}_{n\geq 0}$ and $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1 \leq j \leq K$, satisfy.

Finally, in Section 3.4, we look at the (M,N)-D_ν-coherence relation in a matrix form for regular linear functionals. As a consequence, we get the relation

$$\left(\mathcal{M}_Q - \mathcal{M}_{P,\nu}\right)^2_{\nu^*} = \hbar_{\nu^*}\left[\mathcal{M}_{P,\nu}, \mathcal{M}_Q\right],$$

where $\mathcal{M}_{P,\nu}$ and $\mathcal{M}_Q$ are similar matrices to the monic Jacobi matrices associated with the linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively, $[\mathcal{M}_{P,\nu}, \mathcal{M}_Q]$ is given in (0.0.16) and $(\mathcal{M}_Q - \mathcal{M}_{P,\nu})^2_{\nu^*}$ is the D_{ν^*}-analogue of $(\mathcal{M}_Q - \mathcal{M}_{P,\nu})^2$ defined by

$$(\mathcal{M}_Q - \mathcal{M}_{P,\nu})^2_{\nu^*} = \Big[\mathcal{M}_Q - \mathcal{M}_{P,\nu}\Big]\Big[\mathcal{M}_Q - (\mathcal{M}_{P,\nu} \star \nu^* \mathcal{I})\Big],$$

with $\mathcal{I}$ being the identity matrix and

$$\hbar_\nu = \begin{cases} 1, & \text{if } \nu = \omega, \\ \nu, & \text{if } \nu = q, \end{cases} \qquad \mathcal{J}_P \star \nu \mathcal{I} = \begin{cases} \mathcal{J}_P + \nu \mathcal{I}, & \text{if } \nu = \omega, \\ \nu \mathcal{J}_P, & \text{if } \nu = q. \end{cases}$$

Moreover, when $(\mathcal{U}, \mathcal{V})$ is a (M, N)-D_ν-coherent pair of order m and $\mathcal{U}$ is a D_ν-classical linear functional, it follows that

$$\mathcal{M}_{P^{[m,\nu]}} = \mathcal{M}_Q,$$

where $\mathcal{M}_{P^{[m,\nu]}}$ is a similar matrix to the monic Jacobi matrix associated with the SMOP $\{P_n^{[m,\nu]}(x)\}_{n\geq 0}$.

On the other hand, from a matrix interpretation of the algebraic relation (0.0.18) involving the Sobolev SMOP $\{S_n(x; \lambda, \nu)\}_{n\geq 0}$, we get the following matrix representation of the multiplication operator by x in terms of this basis,

$$x\mathbf{s}(x; \lambda, \nu) = \mathcal{H}_{S,P,\lambda,\nu} \mathbf{s}(x; \lambda, \nu),$$

where $\mathbf{s}(x; \lambda, \nu) = \begin{bmatrix} S_0(x; \lambda, \nu), & S_1(x; \lambda, \nu), & \cdots \end{bmatrix}^T$ and $\mathcal{H}_{S,P,\lambda,\nu}$ is a Hessenberg matrix similar to the monic Jacobi matrix associated with the SMOP $\{P_n(x)\}_{n\geq 0}$.

The following papers contain the work included in this chapter

[12] R. Álvarez-Nodarse, J. Petronilho, N. C. Pinzón-Cortés, and R. Sevinik-Adıgüzel. *On Linearly Related Sequences of Difference Derivatives of Discrete Orthogonal Polynomials.* Submitted.

[76] F. Marcellán and N. C. Pinzón-Cortés. *(1,1)-D_ω-Coherent Pairs.* J. Difference Equ. Appl. **19**, 1828-1848, (2013).

[77] F. Marcellán and N. C. Pinzón-Cortés. *(1,1)-q-Coherent Pairs.* Numer. Algorithms **60** (2), 223-239, (2012).

Chapter 4: Coherent Pairs on the Unit Circle.

In this chapter, we extend the concept of coherent pair to regular Hermitian linear functionals defined on the linear space of Laurent polynomials with complex coefficients and to positive Borel measures supported on the unit circle, by considering the algebraic relation

$$\phi_n^{[m]}(z) + \sum_{i=1}^{M} a_{i,n}\phi_{n-i}^{[m]}(z) = \psi_n(z) + \sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z), \quad n \geq 0,$$
$$\phi_n^{[m]}(z) = \frac{\phi_{n+m}^{(m)}(z)}{(n+1)_m}, \quad m, n \geq 0,$$

between two sequences of monic polynomials on the unit circle $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ orthogonal with respect to the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively, and where the constants M, N, and m are non-negative integers, and $\{a_{i,n}\}_{n\geq 0}$, $0 \leq i \leq M$, and $\{b_{j,n}\}_{n\geq 0}$, $0 \leq j \leq N$, are sequences of complex numbers such that $a_{M,n} \neq 0$, $n \geq M$, $b_{N,n} \neq 0$, $n \geq N$, and $a_{i,n} = b_{i,n} = 0$, $i > n$. In this case, $(\mathcal{U}, \mathcal{V})$ is said to be a *(M, N)-coherent pair of order m on the unit circle*, and if $m = 1$, then $(\mathcal{U}, \mathcal{V})$ is called a *(M, N)-coherent pair on the unit circle*.

In Section 4.1, we analyze the $(1, 1)$-coherence case on the unit circle for regular Hermitian linear functionals. In addition, we determine the cases when $\mathcal{U}$ is either the Lebesgue or the Bernstein-Szegő linear functional and $\mathcal{V}$ is associated with a positive Borel measure on the unit circle, or a rational spectral transformation of it. Thus, we recover the results obtained by A. Branquinho, A. Foulquié Moreno, F. Marcellán, and M. N. Rebocho in [23] for $(1, 0)$-coherent pairs, as well as, by A. Branquinho and M. N. Rebocho in [26] for $(0, 1)$-coherent pairs.

To be more precise, according to the values taken by the first coherence coefficients, we obtain a classification in those situations as follows

⋆ If $\mathcal{U}$ is the Lebesgue linear functional and

- $a_1 = b_1$, then $\mathcal{V}$ is also the Lebesgue linear functional and $a_n = b_n$ for $n \geq 1$.
- $a_2 = a_1 - b_1 \neq 0$ (or, equivalently, $b_r = 0$ for some $r \geq 2$), then $b_n = 0$ and $a_n = a_1 - b_1$ for $n \geq 2$, and $\mathcal{V}$ is the Bernstein-Szegő linear functional with parameter $b_1 - a_1$.
- $a_1, b_1, a_2 \in \mathbb{R}$, and either $0 < a_1 - b_1 < a_2 < 1$ or $-1 < a_2 < a_1 - b_1 < 0$ holds, then the measure associated with $\mathcal{V}$ is given by
$$d\mu_1 = -\frac{b_1 - a_1}{a_2}\frac{1 - |a_2|^2}{|1 + a_2 e^{i\theta}|^2}\frac{d\theta}{2\pi} + \frac{b_1 - a_1 + a_2}{a_2}\frac{d\theta}{2\pi}.$$
- For any values of a_1, b_1 such that $a_1 \neq b_1$, the value of b_2 can be chosen in such a way that $\mathcal{V}$ is the linear functional associated with an anti-associated perturbation of order 1 applied to a measure belonging to the Nevai class.

⋆ If $\mathcal{U}$ is the Bernstein-Szegő linear functional with parameter $-C$, and

- $a_1 = b_1$, then $\mathcal{U}$ and $\mathcal{V}$ are Lebesgue linear functionals, and $a_n = b_n$, $n \geq 1$.
- $a_1 \neq b_1$ and $b_r = 0$ for some $r \geq 3$, then $b_n = 0$ and $a_n = a_1 - b_1$ for $n \geq 2$, $\mathcal{U}$ is the Lebesgue linear functional and $\mathcal{V}$ is the Bernstein-Szegő linear functional with parameter $b_1 - a_1$.
- $a_1 \neq b_1$ and $\frac{1}{2}Ca_2 = b_2\left(a_1 - b_1 + \frac{1}{2}C\right)$, then $\mathcal{V}$ is the Bernstein-Szegő linear functional with parameter $b_1 - a_1 - \frac{1}{2}C$.
- $a_1 \neq b_1$, $\frac{1}{2}Ca_2 \neq b_2\left(a_1 - b_1 + \frac{1}{2}C\right)$ and $b_n \neq 0$ for $n \geq 3$, then b_3 can be chosen so that $\mathcal{V}$ is the linear functional associated with an anti-associated perturbation of order 2 applied to a measure belonging to the Nevai class.

Furthermore, in all previous cases, we give expressions for the coherence coefficients, the moments of $\mathcal{V}$, its orthogonal polynomials, and the Verblunsky coefficients.

In Section 4.2, we study (M, N)-coherent pairs of order m on the unit circle in the framework of Sobolev orthogonal polynomials. More exactly, we investigate the case when the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ forming a (M, N)-coherent pair of order m on the unit circle, constitute the following Soblev inner product

$$\left\langle p(z), q(z)\right\rangle_{\lambda} = \left\langle \mathcal{U}, p(z)\overline{q}(1/z)\right\rangle + \lambda \left\langle \mathcal{V}, p^{(m)}(z)\overline{q^{(m)}}(1/z)\right\rangle, \quad \lambda > 0,\, m \in \mathbb{Z}^{+},$$

where $p(z)$ and $q(z)$ are polynomials with complex coefficients, focusing our attention on the sequence of monic polynomials $\{S_n(z;\lambda)\}_{n\geq 0}$ orthogonal with respect to this inner product. Thereby, we obtain the relation

$$\begin{gathered}\phi_{n+m}(z) + \sum_{i=1}^{M} \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n}\phi_{n-i+m}(z) = S_{n+m}(z;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda} S_{n-j+m}(z;\lambda),\ n \geq 0, \\ S_n(z;\lambda) = \phi_n(z), \quad n \leq m,\end{gathered} \tag{0.0.19}$$

where $K = \max\{M, N\}$, $c_{j,n,\lambda} = 0$ for $n < j \leq K$, and $c_{j,n,\lambda}$, $1 \leq j \leq K$, $n \geq 0$, are rational functions in λ. Conversely, if the previous relation holds, we conclude that $(\mathcal{U}, \mathcal{V})$ must be a (M, K)-coherent pair of order m on the unit circle. (Let us recall that (0.0.19) was proved by A. Branquinho, A. Foulquié Moreno, F. Marcellán, and M. N. Rebocho in [23] for $(1, 0)$-coherent pairs on the unit circle).

Additionally, we get explicit and recursive expressions for $c_{j,n,\lambda}$, $1 \leq j \leq K$, $n \geq 0$, and $\langle S_n(z;\lambda), S_n(z;\lambda)\rangle_{\lambda}$, $n \geq 0$. For some of them it is not necessary to know explicitly the monic Sobolev orthogonal polynomials on the unit circle $S_n(z;\lambda)$, $n \geq 0$. However, we can also obtain the sequence $\{S_n(z;\lambda)\}_{n\geq 0}$ using (0.0.19). Furthermore, the $(1, 1)$-coherence and $(1, 0)$-coherence cases of order m on the unit circle are analyzed in more detail showing additional properties.

To conclude this chapter, in Section 4.3, considering the Hessenberg matrices associated with the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$, more precisely, $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$, matrices similar to those Hessenberg matrices, respectively, we prove that if $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair on the unit circle, then

$$(\mathcal{M}_\phi - \mathcal{M}_\psi)^2 = [\mathcal{M}_\phi, \mathcal{M}_\psi] .$$

where $[\mathcal{M}_\phi, \mathcal{M}_\psi]$ is the commutator of $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ given as (0.0.16). In particular, when $\mathcal{U}$ is the Lebesgue linear functional, we consider the (M, N)-coherence case of order m on the unit circle. In this way, other similar matrix to the Hessenberg matrix associated with $\mathcal{U}$, $\widetilde{\mathcal{M}}_\phi$, satisfies

$$\widetilde{\mathcal{M}}_\phi = \mathcal{M}_\psi.$$

Finally, focussing our attention in the results obtained in Section 4.2, more precisely, in the algebraic relation (0.0.19), when the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ constitute a (M, N)-coherent pair of order m on the unit circle, $m \geq 1$, we give a matrix representation of the multiplication operator by z in terms of the basis of the Sobolev polynomials orthogonal $S_n(z; \lambda)$, $n \geq 0$, involving a lower Hessenberg matrix $\mathcal{H}_{S,\phi,\lambda}$ similar to the Hessenberg matrix associated with the sequence of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$, as follows

$$z\mathbf{s}(z;\lambda) = \mathcal{H}_{S,\phi,\lambda}\mathbf{s}(z;\lambda), \quad \text{where} \quad \mathbf{s}(z;\lambda) = \left[\begin{array}{ccc} S_0(z;\lambda), & S_1(z;\lambda), & \cdots \end{array} \right]^T .$$

The results stated in this chapter have been published in

[38] L. Garza, F. Marcellán, and N. C. Pinzón-Cortés. $(1,1)$-*Coherent Pairs on the Unit Circle.* Abstract Appl. Anal. Accepted. (2013).

[78] F. Marcellán and N. C. Pinzón-Cortés. *Generalized Coherent Pairs on the Unit Circle and Sobolev Orthogonal Polynomials.* Submitted.

Finally, in Chapter 1, **Preliminaries and Notations**, we will recall some basic background from the general theory of orthogonal polynomials that will be needed in the subsequent chapters.

CHAPTER 1

Preliminaries and Notations

In this chapter, we will introduce notations, basic definitions, and the background which will be helpful in the sequel.

1.1 Linear Functionals

We will denote by $\mathbb{P}$ the linear space of polynomials with complex coefficients

$$\mathbb{P} = \text{span}_{\mathbb{C}}\big\{x^n : n \in \mathbb{Z}^+ \cup \{0\}\big\},$$

$\mathbb{P}_n$ stands for the linear subspace of polynomials of degree at most n $n \geq 0$, and $\mathbb{P}^*$ is its algebraic dual space, consisting of all linear functionals $\mathcal{U} : \mathbb{P} \to \mathbb{C}$. Indeed, it is the topological dual space $\mathbb{P}'$ of continuous linear functionals ([85]). Besides, for any linear functional $\mathcal{U}$ and each polynomial $p(x)$

$$\langle \mathcal{U}, p(x)\rangle \text{ will denote the image of } p(x) \text{ by } \mathcal{U}.$$

Every sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$, with $\deg(P_n(x)) = n$, is a basis of $\mathbb{P}$. Then, there exists a unique sequence of linear functionals $\{\mathfrak{p}_n\}_{n\geq 0}$, called the *dual basis of* $\{P_n(x)\}_{n\geq 0}$, such that

$$\langle \mathfrak{p}_n, P_m(x)\rangle = \delta_{n,m}, \quad n, m \geq 0,$$

where $\delta_{n,m}$ denotes the Kronecker delta. As a consequence, a linear functional $\mathcal{U}$ can be expressed as

$$\mathcal{U} = \sum_{n\geq 0} \lambda_n \mathfrak{p}_n, \quad \text{where} \quad \lambda_n = \langle \mathcal{U}, P_n(x)\rangle\,.$$

Furthermore, we can associate to each linear functional $\mathcal{U}$ a sequence of complex numbers $\{u_n\}_{n\geq 0}$ where

$$u_n = \langle \mathcal{U}, x^n \rangle, \quad n \geq 0,$$

which is called the *sequence of moments* of $\mathcal{U}$. Each u_n is said to be the *moment of order* n *of* $\mathcal{U}$, $n \geq 0$.

The *Dirac delta linear functional at* a, $a \in \mathbb{C}$, denoted by δ_a, is defined by

$$\langle \delta_a, p(x) \rangle = p(a), \quad p \in \mathbb{P}.$$

Left multiplication and right multiplication of a linear functional by a polynomial are two operations between polynomials and linear functionals. The first one is a linear functional while the second is a polynomial. They are defined as follows. Let $\mathcal{U}$ and $r(x)$ be a linear functional and a nonzero polynomial, respectively. The linear functionals $r(x)\mathcal{U}$ and $(r(x))^{-1}\mathcal{U}$ are

$$\big\langle r(x)\mathcal{U}, p(x) \big\rangle = \big\langle \mathcal{U}, r(x)p(x) \big\rangle, \quad p \in \mathbb{P},$$
$$\Big\langle \big(r(x)\big)^{-1}\mathcal{U}, p(x) \Big\rangle = \left\langle \mathcal{U}, \frac{p(x) - L_r(x;p)}{r(x)} \right\rangle, \quad p \in \mathbb{P},$$

where $L_r(x;p)$ denotes the (Hermite) interpolation polynomial of $p(x)$ at the zeros of $r(x)$ taking into account their multiplicity. For example, $L_{(x-a)^2}(x;p) = p(a) + p'(a)(x-a)$, for $a \in \mathbb{C}$. On the other hand, $\mathcal{U}\, r(x)$ is the polynomial

$$\mathcal{U}\, r(x) = \left\langle \mathcal{U}_y, \frac{x\, r(x) - y\, r(y)}{x - y} \right\rangle \in \mathbb{P},$$

where $\mathcal{U}_y$ means that the linear functional $\mathcal{U}$ acts on polynomials in the variable y.

Remark 1.1.1. Let $p(x)$ be a nonzero polynomial and let $\mathcal{U}$ be a linear functional. Then

- In $\mathbb{P}'$, the multiplication operators associated with $p(x)$ and $(p(x))^{-1}$ are not commutative. Indeed,

$$(x-a)(x-a)^{-1}\mathcal{U} = \mathcal{U}, \quad \text{but} \quad (x-a)^{-1}(x-a)\mathcal{U} = \mathcal{U} - \langle \mathcal{U}, 1 \rangle\, \delta_a, \quad a \in \mathbb{C}.$$

- If $p_n(x) = \sum_{j=0}^{n} a_{n,j} x^j \in \mathbb{P}_n$, then

$$\mathcal{U} p_n(x) = \sum_{j=0}^{n} a_{n,j} \sum_{k=0}^{j} u_k x^{j-k} = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} a_{n,k} u_{k-j} \right) x^j.$$

Moreover, if we define the polynomial

$$\theta_c p(x) = \frac{p(x) - p(c)}{x - c}, \quad p \in \mathbb{P},\ c \in \mathbb{C}, \tag{1.1.1}$$

then,

$$\big(\mathcal{U}\,\theta_0 p_n\big)(x) = \sum_{j=0}^{n-1} a_{n,j+1} \sum_{k=0}^{j} u_k x^{j-k} = \sum_{j=0}^{n-1} \left(\sum_{k=j}^{n-1} a_{n,k+1} u_{k-j} \right) x^j. \tag{1.1.2}$$

A particular case is the following

Lemma 1.1.2. *Let $\mathcal{U}$ be a linear functional, and let $\varphi(x)$ be a polynomial of degree n whose zeros $x_i \in \mathbb{C}$, $1 \leq i \leq n$, are simple. Then*

$$\big\langle \varphi^{-1}(x)\mathcal{U}, p(x) \big\rangle = \left\langle \mathcal{U}, \frac{p(x) - L_\varphi(x;p)}{\varphi(x)} \right\rangle, \quad p \in \mathbb{P}, \tag{1.1.3}$$

$$\varphi^{-1}(x)\varphi(x)\mathcal{U} = \mathcal{U} - \sum_{i=1}^{n} \frac{1}{\varphi'(x_i)} \left\langle \mathcal{U}, \frac{\varphi(x)}{x - x_i} \right\rangle \delta_{x_i}, \tag{1.1.4}$$

where $L_\varphi(x;p)$ denotes the interpolatory polynomial of $p(x)$ at the zeros of $\varphi(x)$, i.e.,

$$L_\varphi(x;p) = \sum_{i=1}^{n} p(x_i) \frac{\varphi(x)}{(x - x_i)\varphi'(x_i)}.$$

Proof. The proof of (1.1.3) uses induction on n, as follows. Without loss of generality we can assume that $\varphi(x)$ is a monic polynomial, i.e.,

$$\varphi(x) = \prod_{i=1}^{n} (x - x_i),$$

and $p \in \mathbb{P}$. For $n = 1$, $\big\langle (x - x_1)^{-1}\mathcal{U}, p(x) \big\rangle = \left\langle \mathcal{U}, \frac{p(x)-p(x_1)}{x-x_1} \right\rangle$ holds by definition. Now, we assume that (1.1.3) holds, then

$$\begin{aligned} \left\langle \prod_{i=1}^{n+1} (x - x_i)^{-1}\mathcal{U}, p(x) \right\rangle &= \left\langle \prod_{i=1}^{n} (x - x_i)^{-1}\mathcal{U}, \frac{p(x) - p(x_{n+1})}{x - x_{n+1}} \right\rangle \\ &= \left\langle \mathcal{U}, \frac{p(x) - p(x_{n+1})}{\prod_{i=1}^{n+1}(x - x_i)} - \sum_{i=1}^{n} \frac{p(x_i) - p(x_{n+1})}{(x - x_i)\prod_{j=1, j\neq i}^{n+1}(x_i - x_j)} \right\rangle, \end{aligned}$$

which is (1.1.3) for $n+1$, taking into account the following partial fraction decomposition

$$\frac{p(x_{n+1})}{\prod_{i=1}^{n+1}(x - x_i)} = \sum_{i=1}^{n+1} \frac{p(x_{n+1})}{(x - x_i)\prod_{j=1, j\neq i}^{n+1}(x_i - x_j)}.$$

Finally, (1.1.4) follows from (1.1.3). $\square$

On the other hand, the (distributional) *derivative of a linear functional* $\mathcal{U}$ is the linear functional defined by

$$\langle D\mathcal{U}, p(x)\rangle = -\left\langle \mathcal{U}, p'(x)\right\rangle, \quad p \in \mathbb{P}. \tag{1.1.5}$$

Therefore, using induction on n, the mth derivative of the linear functional $\mathcal{U}$, for $m \geq 0$, is given by

$$\langle D^m\mathcal{U}, p(x)\rangle = (-1)^m \left\langle \mathcal{U}, p^{(m)}(x)\right\rangle, \quad p \in \mathbb{P},\ m \geq 0,$$

where $p^{(m)}(x)$ denotes the mth derivative of the polynomial $p(x)$.

Proposition 1.1.3. *Let $\mathcal{U}$ be a linear functional and let $p(x)$ be a polynomial. Then*

$$D^m\left[p(x)\mathcal{U}\right] = \sum_{j=0}^{m} \binom{m}{j} p^{(j)}(x) D^{m-j}\mathcal{U}, \quad m \geq 0. \tag{1.1.6}$$

Proof. It is easy to check this property using induction on m. □

Finally, the *formal Stieltjes series* of a linear functional $\mathcal{U}$ is defined by

$$S_{\mathcal{U}}(z) = -\sum_{n\geq 0} \frac{u_n}{z^{n+1}}\,.$$

In this way, the mth derivative of the formal Stieltjes series of $\mathcal{U}$, $m \geq 0$, is given by

$$S_{\mathcal{U}}^{(m)}(z) = (-1)^{m+1} \sum_{n\geq 0} (n+1)_m \frac{u_n}{z^{n+1+m}}, \tag{1.1.7}$$

where $(n+1)_m$ denotes the *Pochhammer symbol*

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad n \geq 1, \quad \text{and} \quad (a)_0 = 1. \tag{1.1.8}$$

Besides, some useful properties of the Pochhammer symbol are

$$(a+b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_{n-k}(b)_k, \quad \text{and} \quad (-a)_n = (-1)^n(a-n+1)_n. \tag{1.1.9}$$

From now on, if $P_{n+m}(x)$ is a monic polynomial of degree $n+m$, $n \geq 0$, then for a fixed $m \geq 0$, $P_n^{[m]}(x)$ will denote the monic polynomial of degree n

$$P_n^{[m]}(x) = \frac{P_{n+m}^{(m)}(x)}{(n+1)_m}, \quad m, n \geq 0.$$

1.2 The Derivatives D_ω and D_q

W. Hahn ([44]) introduced the linear operator

$$\left(\mathbf{L}_{(q,\omega)}p\right)(x) = \frac{p(qx+\omega)-p(x)}{(q-1)x+\omega}, \quad p \in \mathbb{P},$$

which includes, as particular cases, the *difference operator* D_ω and the *q-derivative operator* D_q given by

$$(D_\omega p)(x) = \left(\mathbf{L}_{(1,\omega)}p\right)(x) = \frac{p(x+\omega)-p(x)}{\omega}, \quad \omega \in \mathbb{C}\setminus\{0\},$$
$$(D_q p)(x) = \left(\mathbf{L}_{(q,0)}p\right)(x) = \frac{p(qx)-p(x)}{(q-1)x} \text{ for } x \neq 0, \quad (D_q p)(0) = p'(0), \quad q \in \mathbb{C}\setminus\{0,1\},$$

respectively. $\mathbf{L}_{(1,1)}$ is the well-known *forward difference operator* Δ and $\mathbf{L}_{(1,-1)}$ is the *backward difference operator* ∇.

We will use the q-numbers

$$[0]_q = 0, \quad [n]_q = \frac{q^n-1}{q-1}, \; n \geq 1.$$

The *q-shifted factorial or q-Pochhammer symbol* $(\alpha;q)_n$, $n \geq 0$, which is the q-analogue of the Pochhammer symbol (1.1.8), is defined by

$$(\alpha;q)_0 = 1, \quad (\alpha;q)_n = (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), \; n \geq 1.$$

The *q-factorial* $[n]_q!$, $n \geq 0$, which is the q-analogue of the *factorial* $n!$, $n \geq 0$, is defined by

$$0! = 1, \quad n! = 1\cdot 2\cdot \ldots \cdot n = (1)_n, \quad n \geq 1,$$
$$[0]_q! = 1, \quad [n]_q! = [1]_q[2]_q\cdots[n]_q = \frac{(q;q)_n}{(1-q)^n}, \quad n \geq 1.$$

Finally, the *Gaussian binomial coefficient or q-binomial coefficient* $\begin{bmatrix} n \\ j \end{bmatrix}_q$, $0 \leq j \leq n$, which is the q-analogue of the *binomial coefficient* $\binom{n}{j}$, $0 \leq j \leq n$, is defined by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \frac{(1)_n}{(1)_j(1)_{n-j}}, \quad 0 \leq j \leq n,$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[j]_q![n-j]_q!} = \frac{(q;q)_n}{(q;q)_j(q;q)_{n-j}}$$

$$= \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-j+1}-1)}{(q-1)(q^2-1)\cdots(q^j-1)}, \quad 0 \le j \le n.$$

Notice that

$$D_q x^n = \frac{q^n x^n - x^n}{(q-1)x} = [n]_q x^{n-1}, \quad n \ge 0,$$

and, as a consequence, for $0 \le m \le n$,

$$\begin{aligned} D_q^m x^n &= [n]_q [n-1]_q \cdots [n-(m-2)]_q [n-(m-1)]_q x^{n-m} \\ &= \frac{q^n-1}{q-1}\frac{q^{n-1}-1}{q-1}\cdots\frac{q^{n-(m-2)}-1}{q-1}\frac{q^{n-(m-1)}-1}{q-1} x^{n-m} = \frac{(q^{n-m+1};q)_m}{(1-q)^m} x^{n-m}. \end{aligned}$$

On the other hand, the leading coefficient of the polynomial of degree $n-m$, $D_\omega^m x^n$, for $0 \le m \le n$, is

$$\text{leadcoeff}\left(D_\omega^m x^n\right) = (n-m+1)_m, \quad 0 \le m \le n.$$

Thus, if $P_{n+m}(x)$ is a monic polynomial of degree $n+m$, $n \ge 0$, then for fixed $m \ge 0$, $P_n^{[m,q]}(x)$ and $P_n^{[m,\omega]}(x)$ will denote the monic polynomials of degree n

$$P_n^{[m,q]}(x) = \frac{D_q^m P_{n+m}(x)}{\frac{(q^{n+1};q)_m}{(1-q)^m}}, \quad P_n^{[m,\omega]}(x) = \frac{D_\omega^m P_{n+m}(x)}{(n+1)_m}, \quad n \ge 0. \tag{1.2.1}$$

From now on,

$$\nu \text{ and } \nu^* \ \text{ will denote either } \ \omega \text{ and } -\omega, \ \text{ or, } \ q \text{ and } q^{-1}, \text{ respectively,}$$

and

$$\eta_{n,m,\nu} = \begin{cases} (n+1)_m, & \text{if } \nu = \omega, \\ \frac{(q^{n+1};q)_m}{(1-q)^m}, & \text{if } \nu = q, \end{cases} \qquad n \ge 0,$$

as a consequence, (1.2.1) becomes

$$P_n^{[m,\nu]}(x) = \frac{D_\nu^m P_{n+m}(x)}{\eta_{n,m,\nu}}, \quad n \ge 0.$$

On the other hand, for a linear functional $\mathcal{U}$, the (distributional) D_ν*-derivative of* $\mathcal{U}$, $D_\nu \mathcal{U}$, is the linear functional given by

$$\langle D_\nu \mathcal{U}, p(x) \rangle = -\langle \mathcal{U}, D_{\nu^*} p(x) \rangle, \quad p \in \mathbb{P}.$$

Remark 1.2.1. Notice that the derivative operator D_ν converges to the usual derivative operator when $q \to 1$ and $\omega \to 0$. Indeed, when $\omega \to 0$ and $q \to 1$, $(D_\nu p)(x)$ converges to $\frac{d}{dx}p(x)$ in $\mathbb{P}$, and, $D_\nu \mathcal{U}$ converges to $D\mathcal{U}$ in $\mathbb{P}^*$ [1], where the linear functional $D\mathcal{U}$ is defined in (1.1.5).

Proposition 1.2.2. *The linear functional $D_\nu^m \mathcal{U}$, for fixed $m \geq 0$, is given by*

$$\langle D_\nu^m \mathcal{U}, p(x)\rangle = (-1)^m \langle \mathcal{U}, D_{\nu^*}^m p(x)\rangle, \quad p \in \mathbb{P}. \tag{1.2.2}$$

Proof. The proof is by induction on m as follows. For $m = 0$ and $m = 1$ it is clear by definition of the linear functional $D_\nu \mathcal{U}$. Now, we assume that (1.2.2) holds for m, then

$$\left\langle D_\nu^{m+1}\mathcal{U}, p(x)\right\rangle = -\langle D_\nu^m \mathcal{U}, D_{\nu^*}p(x)\rangle \overset{(1.2.2)}{=} (-1)^{m+1}\left\langle \mathcal{U}, D_{\nu^*}^{m+1}p(x)\right\rangle.$$

□

Proposition 1.2.3. *For $p, r \in \mathbb{P}$, the following properties hold*

- **(i)** $D_\omega\big[p(x+a)\big] = \big(D_\omega p\big)(x+a), \quad a \in \mathbb{C}$.
- **(ii)** $\big(D_\omega[p\,r]\big)(x) = r(x)\,(D_\omega p)(x) + p(x+\omega)\,(D_\omega r)(x)$.
- **(iii)** $(D_{-\omega}p)(x+\omega) = (D_\omega p)(x)$.
- **(iv)** $D_\omega D_{-\omega} = D_{-\omega} D_\omega$.
- **(v)** $D_\omega^m[p(x)\mathcal{U}] = \sum_{j=0}^m \binom{m}{j}\big(D_\omega^j p\big)\big(x + (m-j)\omega\big)\, D_\omega^{m-j}\mathcal{U}, \quad m \geq 0$.

Proof.
(i)

$$\Big(D_\omega\big[p(x+a)\big]\Big)(x) = \frac{p(x+a+\omega) - p(x+a)}{\omega} = \Big(D_\omega\big[p(x)\big]\Big)(x+a).$$

(ii)

$$\begin{aligned}(D_\omega[p\,r])(x) &= \frac{p(x+\omega)r(x+\omega) - p(x)r(x)}{\omega}\\ &= \frac{p(x+\omega)r(x+\omega) - p(x+\omega)r(x) + p(x+\omega)r(x) - p(x)r(x)}{\omega}\\ &= r(x)\,(D_\omega p)(x) + p(x+\omega)\,(D_\omega r)(x).\end{aligned}$$

(iii)

$$(D_{-\omega}p)(x+\omega) = \frac{p\left((x+\omega)-\omega\right) - p(x+\omega)}{-\omega} = (D_\omega p)(x).$$

[1] $\{\mathcal{U}_n\}_n \subset \mathbb{P}^*$ converges to the linear functional $\mathcal{U}$ if and only if $\{\langle \mathcal{U}_n, p(x)\rangle\}_n \subset \mathbb{C}$ converges to $\langle \mathcal{U}, p(x)\rangle$ for every $p \in \mathbb{P}$.

(iv)

$$
\begin{aligned}
D_\omega D_{-\omega} p(x) = D_\omega \left[\frac{p(x-\omega)-p(x)}{-\omega} \right] &= \frac{\frac{p(x+\omega-\omega)-p(x+\omega)}{-\omega} - \frac{p(x-\omega)-p(x)}{-\omega}}{\omega} \\
&= \frac{\frac{p(x-\omega+\omega)-p(x-\omega)}{\omega} - \frac{p(x+\omega)-p(x)}{\omega}}{-\omega} = D_{-\omega}\left[\frac{p(x+\omega)-p(x)}{\omega} \right] = D_{-\omega} D_\omega p(x),
\end{aligned}
$$

$$
\begin{aligned}
\langle D_\omega D_{-\omega} \mathcal{U}, p(x) \rangle &= -\langle D_{-\omega}\mathcal{U}, D_{-\omega} p(x) \rangle = \langle \mathcal{U}, D_\omega D_{-\omega} p(x) \rangle \\
&= \langle \mathcal{U}, D_{-\omega} D_\omega p(x) \rangle = -\langle D_\omega \mathcal{U}, D_\omega p(x) \rangle = \langle D_{-\omega} D_\omega \mathcal{U}, p(x) \rangle .
\end{aligned}
$$

(v) For the proof we use induction on m. For $m = 0$, it is a trivial result. For $m = 1$, it follows from

$$
\begin{aligned}
\langle D_\omega [p(x)\mathcal{U}], r(x) \rangle &= -\langle p(x)\mathcal{U}, D_{-\omega} r(x) \rangle \overset{s(x-\omega)=p(x)}{=} -\langle \mathcal{U}, s(x-\omega)(D_{-\omega} r)(x) \rangle \\
&= -\big\langle \mathcal{U}, (D_{-\omega}[s\, r])(x) - r(x)(D_{-\omega} s)(x) \big\rangle \\
&= \big\langle D_\omega \mathcal{U}, s(x) r(x) \big\rangle + \big\langle \mathcal{U}, r(x)(D_\omega s)(x-\omega) \big\rangle \\
&\overset{s(x)=p(x+\omega)}{=} \Big\langle p(x+\omega) D_\omega \mathcal{U} + \Big[\big(D_\omega [p(x+\omega)]\big)(x-\omega) \Big] \mathcal{U}, r(x) \Big\rangle \\
&= \Big\langle p(x+\omega) D_\omega \mathcal{U} + \Big[\big(D_\omega [p(x)]\big)(x) \Big] \mathcal{U}, r(x) \Big\rangle .
\end{aligned}
$$

Now, we assume that equality holds for m. Then, for $m+1$ we get

$$
\begin{aligned}
D_\omega^{m+1}[p(x)\mathcal{U}] &= D_\omega \left[\sum_{j=0}^{m} \binom{m}{j} \big(D_\omega^j p\big)\big(x + (m-j)\omega\big) D_\omega^{m-j} \mathcal{U} \right] \\
&= \sum_{j=0}^{m} \binom{m}{j} \Big[\big(D_\omega^j p\big)\big(x + (m-j+1)\omega\big) D_\omega^{m-j+1} \mathcal{U} \\
&\qquad\qquad + D_\omega \big[\big(D_\omega^j p\big)\big(x + (m-j)\omega\big) \big] D_\omega^{m-j} \mathcal{U} \Big] \\
&= \sum_{j=0}^{m} \binom{m}{j} \big(D_\omega^j p\big)\big(x + (m+1-j)\omega\big) D_\omega^{m+1-j} \mathcal{U} \\
&\qquad\qquad + \sum_{k=1}^{m+1} \binom{m}{k-1} \big(D_\omega^k p\big)\big(x + (m-k+1)\omega\big) D_\omega^{m-k+1} \mathcal{U} \\
&= \big(D_\omega^0 p\big)\big(x + (m+1)\omega\big) D_\omega^{m+1} \mathcal{U} + \big(D_\omega^{m+1} p\big)\big(x + (0)\omega\big) D_\omega^0 \mathcal{U} \\
&\qquad + \sum_{j=1}^{m} \left[\binom{m}{j} + \binom{m}{j-1} \right] \big(D_\omega^j p\big)\big(x + (m+1-j)\omega\big) D_\omega^{m+1-j} \mathcal{U}
\end{aligned}
$$

$$= \sum_{j=0}^{m+1} \binom{m+1}{j} \big(D_\omega^j p\big)\big(x + (m+1-j)\omega\big)\, D_\omega^{m+1-j}\mathcal{U}.$$

□

In the following proposition for D_q, we get similar properties to those stated in the previous proposition for D_ω.

Proposition 1.2.4. *For $p, r \in \mathbb{P}$, the following properties hold*

(i) $D_q\big[p(ax)\big] = a\big(D_q p\big)(ax), \quad a \in \mathbb{C} \setminus \{0\}$.
(ii) $\big(D_q\,[p\,r]\big)(x) = r(x)\,(D_q p)\,(x) + p(qx)\,(D_q r)\,(x)$.
(iii) $\big(D_{q^{-1}}p\big)\,(qx) = (D_q p)\,(x)$.
(iv) $D_q D_{q^{-1}} = q^{-1} D_{q^{-1}} D_q$.
(v) $D_q^m\,[p(x)\mathcal{U}] = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_q q^j\,\big(D_q^j p\big)\big(q^{m-j}x\big)\, D_q^{m-j}\mathcal{U}, \quad m \geq 0$.

Proof.
(i)

$$\Big(D_q\big[p(ax)\big]\Big)(x) = \frac{p(qax) - p(ax)}{(q-1)x} = a\Big(D_q\big[p(x)\big]\Big)(ax).$$

(ii)

$$\begin{aligned}(D_q\,[p\,r])\,(x) &= \frac{p(qx)r(qx) - p(x)r(x)}{(q-1)x} = \frac{p(qx)r(qx) - p(qx)r(x) + p(qx)r(x) - p(x)r(x)}{(q-1)x} \\ &= r(x)\,(D_q p)\,(x) + p(qx)\,(D_q r)\,(x).\end{aligned}$$

(iii)

$$\big(D_{q^{-1}}p\big)\,(qx) = \frac{p\big(q^{-1}(qx)\big) - p(qx)}{(q^{-1}-1)\,qx} = (D_q p)\,(x).$$

(iv)

$$\begin{aligned}D_q D_{q^{-1}} p(x) &= D_q\left[\frac{p(q^{-1}x) - p(x)}{(q^{-1}-1)x}\right] = \frac{\frac{p(qq^{-1}x)-p(qx)}{(q^{-1}-1)qx} - \frac{p(q^{-1}x)-p(x)}{(q^{-1}-1)x}}{(q-1)x} \\ &= \frac{\frac{p(q^{-1}x)-p(x)}{(q^{-1}-1)x} - \frac{p(qq^{-1}x)-p(qx)}{(q^{-1}-1)qx}}{(1-q)x} = \frac{\frac{p(q^{-1}qx)-p(q^{-1}x)}{(1-q^{-1})x} - \frac{p(qx)-p(x)}{(1-q^{-1})qx}}{q(q^{-1}-1)x} \\ &= q^{-1}\frac{\frac{p(q^{-1}qx)-p(q^{-1}x)}{(q-1)q^{-1}x} - \frac{p(qx)-p(x)}{(q-1)x}}{(q^{-1}-1)x} = q^{-1} D_{q^{-1}}\left[\frac{p(qx)-p(x)}{(q-1)x}\right] \\ &= q^{-1} D_{q^{-1}} D_q p(x),\end{aligned}$$

$$\begin{aligned}\left\langle D_q D_{q^{-1}}\mathcal{U}, p(x)\right\rangle &= -\left\langle D_{q^{-1}}\mathcal{U}, D_{q^{-1}}p(x)\right\rangle = \left\langle \mathcal{U}, D_q D_{q^{-1}}p(x)\right\rangle \\ &= \left\langle \mathcal{U}, q^{-1}D_{q^{-1}}D_q p(x)\right\rangle = -\left\langle q^{-1}D_q\mathcal{U}, D_q p(x)\right\rangle \\ &= \left\langle q^{-1}D_{q^{-1}}D_q\mathcal{U}, p(x)\right\rangle .\end{aligned}$$

(v) The proof is by induction on m. For $m = 0$, it is immediate. For $m = 1$, we have

$$\begin{aligned}\left\langle D_q\left[p(x)\mathcal{U}\right], r(x)\right\rangle &= -\left\langle p(x)\mathcal{U}, \left(D_{q^{-1}}r\right)(x)\right\rangle \overset{s(q^{-1}x)=p(x)}{=} -\left\langle \mathcal{U}, s(q^{-1}x)\left(D_{q^{-1}}r\right)(x)\right\rangle \\ &= -\left\langle \mathcal{U}, \left(D_{q^{-1}}\left[sr\right]\right)(x) - r(x)\left(D_{q^{-1}}s\right)(x)\right\rangle \\ &= \left\langle D_q\mathcal{U}, r(x)s(x)\right\rangle + \left\langle \mathcal{U}, r(x)\left(D_q s\right)(q^{-1}x)\right\rangle \\ &\overset{s(x)=p(qx)}{=} \left\langle p(qx)D_q\mathcal{U} + \Big(D_q\big[p(qx)\big]\Big)(q^{-1}x)\mathcal{U}, r(x)\right\rangle \\ &= \left\langle p(qx)D_q\mathcal{U} + q\Big(D_q\big[p(x)\big]\Big)(x)\mathcal{U}, r(x)\right\rangle .\end{aligned}$$

Now, we suppose that the equality holds for m, and then we prove it for $m+1$ as follows

$$\begin{aligned}D_q^{m+1}\left[p(x)\mathcal{U}\right] &= D_q\left[\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}_q q^j\left(D_q^j p\right)\left(q^{m-j}x\right)D_q^{m-j}\mathcal{U}\right] \\ &= \sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}_q q^j\Big[\left(D_q^j p\right)\left(q^{m-j+1}x\right)D_q^{m-j+1}\mathcal{U} + qD_q\left[\left(D_q^j p\right)\left(q^{m-j}x\right)\right]\,D_q^{m-j}\mathcal{U}\Big] \\ &= \sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}_q q^j\left(D_q^j p\right)\left(q^{m+1-j}x\right)D_q^{m+1-j}\mathcal{U} \\ &\qquad + \sum_{k=1}^{m+1}\begin{bmatrix} m \\ k-1 \end{bmatrix}_q q^{k-1}qq^{m-k+1}\left(D_q^k p\right)\left(q^{m-k+1}x\right)D_q^{m-k+1}\mathcal{U} \\ &= \left(D_q^0 p\right)\left(q^{m+1}x\right)D_q^{m+1}\mathcal{U} + q^{m+1}\left(D_q^{m+1}p\right)\left(q^0 x\right)D_q^0\mathcal{U} \\ &\quad + \sum_{j=1}^{m}\left(\begin{bmatrix} m \\ j \end{bmatrix}_q + \begin{bmatrix} m \\ j-1 \end{bmatrix}_q q^{m+1-j}\right)q^j\left(D_q^j p\right)\left(q^{m+1-j}x\right)D_q^{m+1-j}\mathcal{U} \\ &= \sum_{j=0}^{m+1}\begin{bmatrix} m+1 \\ j \end{bmatrix}_q q^j\left(D_q^j p\right)\left(q^{m+1-j}x\right)D_q^{m+1-j}\mathcal{U}.\end{aligned}$$

□

Remark 1.2.5. In the sequel, the following notation will be used

$$\hbar_\nu = \begin{cases} 1, & \text{if } \nu = \omega, \\ \nu, & \text{if } \nu = q, \end{cases} \qquad x \star \nu = \begin{cases} x+\nu, & \text{if } \nu = \omega, \\ \nu x, & \text{if } \nu = q. \end{cases}$$

Hence, Propositions 1.2.3 and 1.2.4 can be rewritten as, for $p, r \in \mathbb{P}$,

(i.a) $D_\omega\big[p(x+a)\big] = \big(D_\omega p\big)(x+a), \quad a \in \mathbb{C}.$
(i.b) $D_q\big[p(ax)\big] = a\big(D_q p\big)(ax), \quad a \in \mathbb{C}\setminus\{0\}.$
(ii) $\big(D_\nu\,[p\,r]\big)(x) = r(x)\big(D_\nu p\big)(x) + p(x\star\nu)\big(D_\nu r\big)(x).$
(iii) $\big(D_{\nu^*} p\big)(x\star\nu) = \big(D_\nu p\big)(x).$
(iv) $D_\nu D_{\nu^*} = \hbar_{\nu^*} D_{\nu^*} D_\nu.$
(v) $D_\nu^m\big[p(x)\mathcal{U}\big] = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_\nu \hbar_\nu^j \big(D_\nu^j p\big)\big(x\underbrace{\star\nu\cdots\star\nu}_{m-j \text{ times}}\big)\, D_\nu^{m-j}\mathcal{U}, \quad m \geq 0.$

Here $\begin{bmatrix} m \\ j \end{bmatrix}_\omega = \binom{m}{j}$.

1.3 Orthogonal Polynomials

For a linear functional $\mathcal{U}$, we can consider the Hankel matrix

$$H = \big[u_{i+j}\big]_{i,j=0}^{\infty}$$

associated with its moments. $\mathcal{U}$ said to be a *quasi-definite or regular* linear functional (see [31]) if all principal leading submatrices are nonsingular, i.e.,

$$\det(H_n) = \det\Big(\big[u_{i+j}\big]_{i,j=0}^{n}\Big) = \begin{vmatrix} u_0 & u_1 & \cdots & u_n \\ u_1 & u_2 & \cdots & u_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_n & u_{n+1} & \cdots & u_{2n} \end{vmatrix} \neq 0, \quad n \geq 0.$$

This condition is equivalent to the existence of a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ such that

i) $\deg(P_n(x)) = n$, for all $n \geq 0$,

ii) $\langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n^P\,\delta_{n,m}\,, \quad k_n \neq 0 \quad \text{and} \quad n, m \geq 0.$

$\{P_n(x)\}_{n\geq 0}$ is said to be a *sequence of orthogonal polynomials (SOP)* with respect to the linear functional $\mathcal{U}$. This sequence is unique up to multiplicative constants. As a consequence, if all polynomials of the sequence are monic, $\{P_n(x)\}_{n\geq 0}$ is called the *sequence of monic orthogonal polynomials (SMOP)* with respect to the linear functional $\mathcal{U}$.

When all principal leading submatrices of the Hankel matrix H are positive definite, i.e., $\det(H_n) > 0$, $n \geq 0$, then $\mathcal{U}$ is said to be a *positive definite* linear functional. In this case, there exists a positive Borel measure μ supported on an infinite subset E of the real line such that

$$\langle \mathcal{U}, p(x)\rangle = \int_E p(x)d\mu(x), \quad p \in \mathbb{P}.$$

Besides, the following inequality holds

$$\int_E P_n^2(x)d\mu(x) < \int_E p^2(x)d\mu(x),$$

for every monic polynomial of degree n, $p(x) \neq P_n(x)$, which means the extremal character of the SMOP $\{P_n(x)\}_{n\geq 0}$ (see e.g. [112]).

An important characterization of orthogonal polynomials and regular linear functionals is given by the Favard Theorem as follows

Theorem 1.3.1 (Favard Theorem, [31]). *A sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ is a SMOP with respect to a regular linear functional $\mathcal{U}$ (which is unique if $u_0 = 1$) if and only if there exist sequences of complex numbers $\{\alpha_n^P\}_{n\geq 0}$ and $\{\beta_n^P\}_{n\geq 0}$, $\beta_n^P \neq 0$, $n \geq 2$, such that they satisfy a* three-term recurrence relation (TTRR)

$$\begin{aligned} P_n(x) &= \left(x - \alpha_n^P\right) P_{n-1}(x) - \beta_n^P P_{n-2}(x), \quad n \geq 2, \\ P_1(x) &= x - \alpha_1^P, \quad P_0(x) = 1, \end{aligned} \tag{1.3.1}$$

where

$$\alpha_n^P = \frac{\langle \mathcal{U}, xP_{n-1}^2(x)\rangle}{\langle \mathcal{U}, P_{n-1}^2(x)\rangle}, \quad \beta_{n+1}^P = \frac{\langle \mathcal{U}, P_n^2(x)\rangle}{\langle \mathcal{U}, P_{n-1}^2(x)\rangle} \neq 0, \quad n \geq 1.$$

Moreover, the linear functional $\mathcal{U}$ is positive definite if and only if α_n^P is real and $\beta_{n+1}^P > 0$, for $n \geq 1$.

Remark 1.3.2. The TTRR given by (1.3.1) can be written in a matrix form as

$$x\mathbf{p}(x) = \mathcal{J}_P \mathbf{p}(x), \tag{1.3.2}$$

where

$$\mathbf{p}(x) = \left[\begin{array}{ccc} P_0(x), & P_1(x), & \cdots \end{array}\right]^T, \quad \mathcal{J}_P = \begin{bmatrix} \alpha_1^P & 1 & 0 & 0 & \cdots \\ \beta_2^P & \alpha_2^P & 1 & 0 & \ddots \\ 0 & \beta_3^P & \alpha_3^P & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where the semi-infinite tridiagonal matrix $\mathcal{J}_P$ is said to be the *monic Jacobi matrix* associated with the regular linear functional $\mathcal{U}$.

On the other hand, every monic orthogonal polynomial $P_n(x)$, $n \geq 0$, with respect to a regular linear functional $\mathcal{U}$ can be written as (*Heine's formula*)

$$P_n(x) = \frac{1}{\det(H_{n-1})} \begin{vmatrix} u_0 & u_1 & \cdots & u_{n-1} & u_n \\ u_1 & u_2 & \cdots & u_n & u_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n-1} & u_n & \cdots & u_{2n-2} & u_{2n-1} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad n \geq 1, \qquad P_0(x) = 1.$$

Furthermore, any polynomial $r_n(x)$ of degree n, $n \geq 0$, can be expanded in terms of the SMOP $\{P_n(x)\}_{n\geq 0}$ as follows

$$r_n(x) = \sum_{j=0}^{n} \zeta_{j,n} P_j(x), \quad \text{where} \quad \zeta_{j,n} = \frac{\langle \mathcal{U}, r_n(x)P_j(x)\rangle}{\langle \mathcal{U}, P_j^2(x)\rangle},$$

When $\mathcal{U}$ is positive definite, $\zeta_{0,n}, \ldots, \zeta_{n,n}$ are the corresponding Fourier coefficients of $r_n(x)$.

The following proposition presents some results relating the dual basis of a SMOP and its respective linear functional and derivatives.

Proposition 1.3.3. *Let $\{P_n(x)\}_{n\geq 0}$ be the SMOP with respect to a regular linear functional $\mathcal{U}$ and let $\{\mathfrak{p}_n\}_{n\geq 0}$ be its dual basis. Then*

$$\mathfrak{p}_n = \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x)\rangle} \mathcal{U}, \quad n \geq 0. \tag{1.3.3}$$

Besides, for a fixed $m \geq 0$,

$$D^m \mathfrak{e}_{n,m} = (-1)^m (n+1)_m \mathfrak{p}_{n+m}, \quad n \geq 0, \tag{1.3.4}$$

where $\{\mathfrak{e}_{n,m}\}_{n\geq 0}$ is the dual basis of the sequence of monic polynomials $\{P_n^{[m]}(x)\}_{n\geq 0}$.

As a consequence,

$$D^m \mathfrak{e}_{n,m} = (-1)^m \frac{(n+1)_m}{\langle \mathcal{U}, P_{n+m}^2(x)\rangle} P_{n+m}(x)\mathcal{U}, \quad n \geq 0.$$

Proof. The proof is similar to the proof of the following proposition where we will use the derivative operator D_ν instead of D. □

Proposition 1.3.4. *Let $\{P_n(x)\}_{n\geq 0}$ be the SMOP with respect to a linear functional $\mathcal{U}$, let $\{\mathfrak{p}_n\}_{n\geq 0}$ be its corresponding dual basis, and let $\{\mathfrak{e}_{n,m,\nu}\}_{n\geq 0}$ be the dual basis of the sequence of monic polynomials $\{P_n^{[m,\nu]}(x)\}_{n\geq 0}$, for fixed $m \geq 0$. Then,*

$$\mathfrak{p}_n = \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x)\rangle} \mathcal{U} \quad \text{and} \quad D_{\nu^*}^m \mathfrak{e}_{n,m,\nu} = (-1)^m \eta_{n,m,\nu} \mathfrak{p}_{n+m}, \quad n \geq 0, \tag{1.3.5}$$

and, as a consequence,

$$D_{\nu^*}^m \mathfrak{e}_{n,m,\nu} = (-1)^m \eta_{n,m,\nu} \frac{P_{n+m}(x)}{\langle \mathcal{U}, P_{n+m}^2(x)\rangle} \mathcal{U}, \quad n, m \geq 0.$$

Proof. Notice that,

$$P_n(x)\mathcal{U} = \sum_{k\geq 0} \langle P_n(x)\mathcal{U}, P_k(x)\rangle \mathfrak{p}_k = \langle \mathcal{U}, P_n^2(x)\rangle \mathfrak{p}_n,$$

and,

$$\begin{aligned} D_{\nu^*}^m \mathfrak{e}_{n,m,\nu} &= \sum_{j\geq 0} \langle D_{\nu^*}^m \mathfrak{e}_{n,m,\nu}, P_j(x)\rangle \mathfrak{p}_j = \sum_{k\geq -m} (-1)^m \langle \mathfrak{e}_{n,m,\nu}, D_\nu^m P_{k+m}(x)\rangle \mathfrak{p}_{k+m} \\ &= (-1)^m \sum_{k\geq 0} \left\langle \mathfrak{e}_{n,m,\nu}, \eta_{k,m,\nu} P_k^{[m,\nu]}(x)\right\rangle \mathfrak{p}_{k+m} = (-1)^m \eta_{n,m,\nu}\mathfrak{p}_{n+m}. \end{aligned}$$

□

In the next proposition, we state the relation between the measures associated with two positive definite linear functionals which are related by a rational factor.

Proposition 1.3.5. *Let $\mathcal{U}$ and $\mathcal{V}$ be two positive definite linear functionals such that*

$$\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}, \tag{1.3.6}$$

where $\varphi(x)$ and $\rho(x)$ are nonzero polynomials of degree r and t, respectively, and let μ_0 and μ_1 be their corresponding positive Borel measures supported on the real line. We assume that μ_1 has compact support and all zeros of $\varphi(x)$ are real and simple and they lie out the convex-hull of the support of μ_1, i.e., $x_i \in \mathbb{R}\setminus \mathrm{co}(\mathrm{supp}(\mu_1))$ for all $1 \leq i \leq r$. Additionally, for each $\ell = 1, \cdots, r$, we define

$$\begin{aligned} \eta_\ell = \frac{1}{\varphi'(x_\ell)} \int_{\mathbb{R}} \frac{\varphi(x)}{x - x_\ell} d\mu_0 - \frac{1}{r\varphi'(x_\ell)} \sum_{i=1}^{r} \Bigg\{ \rho(x_i) F(x_\ell, \mu_1) \\ + \sum_{j=0}^{t-1} \frac{\left(\theta_{x_i}\rho(x)\right)^{(j)}(0)}{j!} \left[v_j + (x_\ell - x_i) \sum_{k=0}^{j-1} x_\ell^{j-1-k} v_k + x_\ell^j (x_\ell - x_i) F(x_\ell, \mu_1) \right] \Bigg\}, \end{aligned}$$

where $\theta_{x_i}\rho(x)$ is given by (1.1.1), and $F(\cdot, \mu_1)$ is the Cauchy transform of the measure μ_1 defined by

$$F(z, \mu_1) = \int_{\mathbb{R}} \frac{d\mu_1(x)}{x - z}, \quad z \in \mathbb{C}\setminus \mathrm{co}(\mathrm{supp}(\mu_1)).$$

Then, the measures μ_0 and μ_1 satisfy

$$d\mu_0(x) = \frac{\rho(x)}{\varphi(x)} d\mu_1(x) + \sum_{\ell=1}^{r} \eta_\ell \delta_{x_\ell}, \tag{1.3.7}$$

provided $\eta_\ell \geq 0$ for all $\ell = 1, \cdots, r$ and the right hand side of (1.3.7) defines a positive Borel measure.

Proof. From (1.3.6) we have that $\varphi^{-1}(x)\varphi(x)\mathcal{U} = \varphi^{-1}(x)\rho(x)\mathcal{V}$. Hence, from (1.1.3) we get, for every fixed $p \in \mathbb{P}$,

$$\left\langle \mathcal{U}, \frac{p(x) - L_\varphi(x;p)}{\varphi(x)}\varphi(x) \right\rangle = \left\langle \mathcal{V}, \frac{p(x) - L_\varphi(x;p)}{\varphi(x)}\rho(x) \right\rangle.$$

Consequently,

$$\int_{\mathbb{R}} p(x)d\mu_0 - \int_{\mathbb{R}} L_\varphi(x;p)d\mu_0 = \int_{\mathbb{R}} p(x)\frac{\rho(x)}{\varphi(x)}d\mu_1 - \int_{\mathbb{R}} L_\varphi(x;p)\frac{\rho(x)}{\varphi(x)}d\mu_1,$$

where, since $L_\varphi(x;p) = \sum_{i=1}^r p(x_i)\frac{\varphi(x)}{(x-x_i)\varphi'(x_i)}$, we get

$$\int_{\mathbb{R}} L_\varphi(x;p)d\mu_0 - \int_{\mathbb{R}} L_\varphi(x;p)\frac{\rho(x)}{\varphi(x)}d\mu_1 = \sum_{\ell=1}^{r}\frac{p(x_\ell)}{\varphi'(x_\ell)}\left[\int_{\mathbb{R}}\frac{\varphi(x)}{x-x_\ell}d\mu_0 - \int_{\mathbb{R}}\frac{\rho(x)}{x-x_\ell}d\mu_1\right].$$

On the other hand, since $\theta_{x_i}\rho(x)$ is a polynomial of degree $t-1$, then

$$\rho(x) = \frac{1}{r}\sum_{i=1}^{r}\rho(x) = \frac{1}{r}\sum_{i=1}^{r}\left[\left(\sum_{j=0}^{t-1}\frac{\left(\theta_{x_i}\rho(x)\right)^{(j)}(0)}{j!}x^j\right)(x-x_i) + \rho(x_i)\right].$$

Thus, the proof is complete taking into account that

$$\frac{x^j(x-x_i)}{x-x_\ell} = x^j + (x_\ell - x_i)\sum_{k=0}^{j-1}x_\ell^{j-1-k}x^k + \frac{x_\ell^j(x_\ell - x_i)}{x-x_\ell}.$$

□

Remark 1.3.6. Under the remaining hypothesis of Proposition 1.3.5, the right-hand side of (1.3.7) defines a positive Borel measure if the polynomials $\rho(x)$ and $\varphi(x)$ have the same sign in the interval $\mathrm{co}(\mathrm{supp}(\mu_1))$.

1.3.1 Weakly Quasi-Definite Linear Functionals

A linear functional $\mathcal{U}$ is said to be *weakly quasi-definite of order* Υ, $\Upsilon \in \mathbb{N}\cup\{\infty\}$, if the leading principal submatrices H_n of the Hankel matrix associated with the moments of the linear functional are nonsingular for $0 \le n \le \Upsilon$ and, if $\Upsilon < \infty$, $H_{\Upsilon+1}$ is a singular matrix. As a consequence, there exists a countable family of polynomials $\{P_n(x)\}_{n=0}^{\Upsilon}$ called the *family of monic orthogonal polynomials (MOP)* with respect to $\mathcal{U}$, such that

$$\deg(P_n(x)) = n \text{ and } \langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n\delta_{n,m},\ k_n \neq 0,\ 0 \le n,m \le \Upsilon.$$

Besides, this family of MOP satisfies the following TTRR

$$\begin{gathered} P_n(x) = \left(x - \alpha_n^P\right) P_{n-1}(x) - \beta_n^P P_{n-2}(x), \quad \beta_n^P \neq 0, \quad 1 \leq n \leq \Upsilon, \\ P_0(x) = 1, \quad P_{-1}(x) = 0. \end{gathered} \tag{1.3.8}$$

Conversely, if a family of monic polynomials $\{P_n(x)\}_{n=0}^{\Upsilon}$ satisfies (1.3.8), then $\{P_n(x)\}_{n=0}^{\Upsilon-1}$ is orthogonal with respect to a weakly quasi-definite linear functional.

Remark 1.3.7.

- When $\mathcal{U}$ is a weakly quasi-definite linear functional of order Υ with $\Upsilon < \infty$, then there exists a unique family of monic polynomials $\{P_n(x)\}_{n=0}^{\Upsilon+1}$ such that

  $$\begin{gathered} \langle \mathcal{U}, x^m P_n(x) \rangle = 0, \quad 0 \leq m \leq n-1, \quad 1 \leq n \leq \Upsilon + 1, \\ \langle \mathcal{U}, x^n P_n(x) \rangle \neq 0, \quad 0 \leq n \leq \Upsilon, \quad \text{and} \quad \left\langle \mathcal{U}, x^{\Upsilon+1} P_{\Upsilon+1}(x) \right\rangle = 0, \end{gathered}$$

  and, therefore, $\{P_n(x)\}_{n=0}^{\Upsilon}$ is the family of MOP associated with $\mathcal{U}$.

- If $\Upsilon = \infty$, a weakly quasi-definite linear functional is a regular linear functional.

- Given a family of monic polynomials $\{P_n(x)\}_{n=0}^{\Upsilon}$ with $\deg(P_n(x)) = n, 0 \leq n \leq \Upsilon$, and $\Upsilon \in \mathbb{N} \cup \{\infty\}$, there exists a family of linear functionals $\{\mathfrak{p}_n\}_{n=0}^{\Upsilon}$ called the *dual family* of $\{P_n(x)\}_{n=0}^{\Upsilon}$ such that

  $$\langle \mathfrak{p}_j, P_k(x) \rangle = \delta_{j,k}, \quad 0 \leq j, k \leq \Upsilon.$$

  In this way, if $\{P_n(x)\}_{n=0}^{\Upsilon}$ is the family of MOP associated with a weakly quasi-definite linear functional $\mathcal{U}$ of order Υ, then the equations in (1.3.5) read

  $$\begin{gathered} \mathfrak{p}_n = \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x) \rangle} \mathcal{U}, \quad 0 \leq n \leq \Upsilon, \\ D_{\nu^*}^m \mathfrak{e}_{n,m,\nu} = (-1)^m \eta_{n,m,\nu} \mathfrak{p}_{n+m}, \quad 0 \leq n \leq \Upsilon - m, \end{gathered} \tag{1.3.9}$$

  where m is a fixed non-negative integer and $\{\mathfrak{e}_{n,m,\nu}\}_{n=0}^{\Upsilon-m}$ is the family dual of the monic polynomials $\{P_n^{[m,\nu]}(x)\}_{n=0}^{\Upsilon-m}$.

1.4 Semiclassical Linear Functionals

Let $\sigma(x)$ and $\tau(x)$ be two nonzero polynomials such that

$$\sigma(x) = a_\ell x^\ell + \dots, \; a_\ell \neq 0, \; \ell \geq 0, \quad \text{and} \quad \tau(x) = b_t x^t + \dots, \; b_t \neq 0, \; t \geq 1.$$

$(\sigma(x), \tau(x))$ is said to be an *admissible pair* if

$$\text{either} \quad \ell - 1 \neq t \quad \text{or} \quad \ell - 1 = t \;\text{ and }\; n a_{t+1} + b_t \neq 0, \; n \geq 0.$$

When a linear functional $\mathcal{U}$ is regular and there exists an admissible pair of nonzero polynomials $(\sigma(x), \tau(x))$, $\sigma(x)$ monic and $\deg(\tau(x)) \geq 1$, such that the following distributional Pearson equation holds

$$D(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}, \tag{1.4.1}$$

then $\mathcal{U}$ is said to be a *semiclassical* linear functional ([86]). (1.4.1) implies that $\sigma(x)$ can not be zero and $\tau(x)$ can not be a constant. Otherwise, $\mathcal{U}$ would lose its regular condition since $H_0 = u_0 = 0$.

The *class* of a semiclassical linear functional $\mathcal{U}$ is the non-negative integer

$$s = \min \max\big\{\deg(\sigma(x)) - 2, \deg(\tau(x)) - 1\big\},$$

where the minimum is taken among all admissible pairs of nonzero polynomials $(\sigma(x), \tau(x))$ which satisfy (1.4.1). This value is defined as a minimum because, for instance, if the admissible pair $(\sigma(x), \tau(x))$ satisfies (1.4.1), then for every nonzero polynomial $p(x)$, so does $\big(p(x)\sigma(x),\, p(x)\tau(x) + p'(x)\sigma(x)\big)$, which is also an admissible pair. As a consequence, such a maximum can be as large as desired. On the other hand, the admissible pair determining the class is unique assuming $\sigma(x)$ is monic.

The SMOP associated with a semiclassical linear functional of class s is called *semiclassical SMOP of class s*.

The following proposition allows us to deduce the class of a semiclassical linear functional.

Proposition 1.4.1 ([86]). *Let $\mathcal{U}$ a semiclassical linear functional given by (1.4.1). The class of $\mathcal{U}$ is s if and only if some of the following statements holds*

i. *The polynomials $\sigma(x)$ and $\tau(x) - \sigma'(x)$ are coprime.*

ii. *If c is a common zero of $\sigma(x)$ and $\tau(x) - \sigma'(x)$, then*

$$\big\langle \mathcal{U}, \widetilde{\tau}_c(x) + \sigma_c'(x) \big\rangle \neq 0,$$

where

$$\sigma(x) = (x - c)\sigma_c(x) \quad \text{and} \quad \tau(x) - \sigma'(x) = (x - c)\widetilde{\tau}_c(x).$$

The previous conditions can be written as

$$\prod_{\{c \in \mathbb{C} : \sigma(c) = 0\}} \left(\Big|\tau(c) - \sigma'(c)\Big| + \Big|\big\langle \mathcal{U}, \widetilde{\tau}_c(x) + \sigma_c'(x) \big\rangle\Big| \right) > 0,$$

or, equivalently,

$$\prod_{\{c \in \mathbb{C} : \sigma(c) = 0\}} \left(\Big|\tau(c) - \sigma'(c)\Big| + \Big|\big\langle \mathcal{U}, \theta_c \tau(x) - \theta_c^2 \sigma(x) \big\rangle\Big| \right) > 0.$$

Next, we present some properties of semiclassical linear functionals. For more characterizations, see for instance [20, 21, 22, 86, 88, 82, 83, 96] and the references therein.

Proposition 1.4.2 ([86]). *A regular linear functional $\mathcal{U}$ is semiclassical if and only if there exist polynomials $\sigma(z)$, $C(z)$, and $D(z)$ such that its formal Stieltjes series $S_{\mathcal{U}}$ is a (formal) solution of the following non-homogeneous first order linear differential equation*

$$\sigma(z)S'_{\mathcal{U}}(z) = C(z)S_{\mathcal{U}} + D(z). \tag{1.4.2}$$

Moreover, if the polynomials $\sigma(z)$, $C(z)$, and $D(z)$ are mutually coprime, then the class of $\mathcal{U}$ is given by

$$s = \max\big\{\deg(C(z)) - 1, \deg(D(z))\big\}.$$

Remark 1.4.3. Up to common factors, the polynomial $\sigma(z)$ in (1.4.2) is the same in (1.4.1), and the polynomials $C(z)$ and $D(z)$ are as

$$C(z) = \tau(z) - \sigma'(z) \quad \text{and} \quad D(z) = \big(\mathcal{U}\theta_0\tau\big)(z) - \big(\mathcal{U}\theta_0\sigma\big)'(z).$$

Proposition 1.4.4. *If two regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ are related by an expression of rational type as*

$$p(x)\mathcal{U} = r(x)\mathcal{V},$$

where $p(x)$ and $r(x)$ are nonzero polynomials, then, $\mathcal{U}$ is a semiclassical linear functional if and only if so is $\mathcal{V}$. Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s + \deg(p(x)) + \deg(r(x))$.

Proof. If $\mathcal{U}$ satisfies (1.4.1), then $\mathcal{V}$ satisfies

$$D\big[p(x)r(x)\sigma(x)\mathcal{V}\big] = \big[p(x)\tau(x) + 2p'(x)\sigma(x)\big]r(x)\mathcal{V}.$$

The other statements can be deduced in a straightforward way. □

1.4.1 D_ν-Semiclassical Linear Functionals

The D_ν-semiclassical linear functional concept was introduced by P. Maroni in [86, p. 128]. A linear functional $\mathcal{U}$ is called *D_ν-semiclassical* if it is weakly quasi-definite and there exist a monic polynomial $\sigma(x)$ and a polynomial $\tau(x)$ of degree ≥ 1 such that $\mathcal{U}$ satisfies the distributional equation (D_ν-Pearson equation)

$$D_\nu\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U}. \tag{1.4.3}$$

In these conditions, the *class of $\mathcal{U}$* is defined by the non-negative integer

$$s = \min\max\{\deg(\sigma(x)) - 2, \deg(\tau(x)) - 1\},$$

where the minimum is taken among all pairs of polynomials $(\sigma(x), \tau(x))$ such that (1.4.3) holds. When $\sigma(x)$ is monic, the pair that determines the class is unique. We also say that the family of MOP associated with $\mathcal{U}$ is *D_ν-semiclassical of class s* if the class of $\mathcal{U}$ is s.

Remark 1.4.5. Notice that

- (1.4.3) implies that $\sigma(x)$ can not be zero and $\tau(x)$ can not be a constant. Otherwise, $\mathcal{U}$ would not be weakly quasi-definite since $H_0 = u_0 = 0$.
- The class of a D_ν-semiclassical linear functional is defined as a minimum because the set of pairs, except for multiplicative constants, which satisfy (1.4.3) is not a singleton, and that value can be as large as desired. Indeed, if $(\sigma(x), \tau(x))$ satisfies (1.4.3), then so does

$$\Big(p(x \star \nu^*)\sigma(x),\, p(x)\tau(x) + (D_\nu p)(x \star \nu^*)\sigma(x)\Big), \quad \text{for every } p(x) \in \mathbb{P} \setminus \{0\}.$$

The following result provides a criterion for determining the class of a D_ν-semiclassical linear functional.

Theorem 1.4.6 ([86], [89] for $\nu = \omega$, and, [57] for $\nu = q$)**.** *Let $\mathcal{U}$ be a D_ν-semiclassical linear functional satisfying (1.4.3). Then, the class of $\mathcal{U}$ is s if and only if*

$$\prod_{\{c \in \mathbb{C} : \sigma(c) = 0\}} \Big[\Big| \hbar_{\nu^*} \tau(c \star \nu^*) - \big(D_{\nu^*}\sigma\big)(c) \Big| + \Big| \big\langle \mathcal{U}, \hbar_{\nu^*} \big(\theta_{c\star\nu^*}\tau\big)(x) - \big(\theta_{c\star\nu^*} \circ \theta_c \sigma\big)(x) \big\rangle \Big| \Big] > 0,$$

holds. If there exists $c \in \mathbb{C}$ such that $\sigma(c) = 0$ and

$$\hbar_{\nu^*}\tau(c \star \nu^*) - \big(D_{\nu^*}\sigma\big)(c) = \big\langle \mathcal{U}, \hbar_{\nu^*} \big(\theta_{c\star\nu^*}\tau\big)(x) - \big(\theta_{c\star\nu^*} \circ \theta_c \sigma\big)(x) \big\rangle = 0,$$

then the D_ν-Pearson equation (1.4.3) becomes

$$D_\nu \big[\theta_c \sigma(x) \mathcal{U}\big] = \Big[\hbar_{\nu^*} \big(\theta_{c\star\nu^*}\tau\big)(x) - \big(\theta_{c\star\nu^*} \circ \theta_c \sigma\big)(x)\Big]\mathcal{U}.$$

Next, we state some characterizations of D_ν-semiclassical linear functionals.

Proposition 1.4.7. *Let $\mathcal{U}$ be a linear functional. The following equivalences hold*

$$D_\omega [\sigma(x)\mathcal{U}] = \tau(x)\mathcal{U} \quad \Longleftrightarrow \quad D_{-\omega}\Big(\big[\sigma(x) + \omega\tau(x)\big]\mathcal{U}\Big) = \tau(x)\mathcal{U},$$
$$D_q [\sigma(x)\mathcal{U}] = \tau(x)\mathcal{U} \quad \Longleftrightarrow \quad D_{q^{-1}}\Big(\big[q\sigma(x) + (q-1)x\tau(x)\big]\mathcal{U}\Big) = \tau(x)\mathcal{U}.$$

Thus, $\mathcal{U}$ is D_ν-semiclassical if and only if it is D_{ν^*}-semiclassical. Moreover, $\mathcal{U}$ is D_ν-semiclassical of class s if and only if it is D_{ν^*}-semiclassical of class s.

Proof. Since for every $p \in \mathbb{P}$,

$$\begin{aligned} \langle D_\omega [\sigma(x)\mathcal{U}] - \tau(x)\mathcal{U}, p(x+\omega)\rangle &= -\langle \mathcal{U}, \sigma(x) D_{-\omega}[p(x+\omega)] + \tau(x)p(x+\omega)\rangle \\ &= -\big\langle \mathcal{U}, \sigma(x) D_\omega p(x) + \tau(x)\big[p(x) + \omega D_\omega p(x)\big]\big\rangle \end{aligned}$$

$$= \left\langle D_{-\omega}\Big(\big[\sigma(x)+\omega\tau(x)\big]\mathcal{U}\Big) - \tau(x)\mathcal{U}, p(x)\right\rangle,$$

$$\begin{aligned}
\langle D_q\left[\sigma(x)\mathcal{U}\right] - \tau(x)\mathcal{U}, p(qx)\rangle &= -\left\langle \mathcal{U}, \sigma(x)D_{q^{-1}}\left[p(qx)\right] + \tau(x)p(qx)\right\rangle \\
&= -\left\langle \mathcal{U}, q\sigma(x)D_q p(x) + \tau(x)\big[p(x) + (q-1)xD_q p(x)\big]\right\rangle \\
&= \left\langle D_{q^{-1}}\Big(\big[q\sigma(x)+(q-1)x\tau(x)\big]\mathcal{U}\Big) - \tau(x)\mathcal{U}, p(x)\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&D_\nu\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U} \quad \Longleftrightarrow \quad \langle D_\nu\left[\sigma(x)\mathcal{U}\right] - \tau(x)\mathcal{U}, p(x)\rangle = 0,\ \forall p\in\mathbb{P}, \\
&D_{-\omega}\Big(\big[\sigma(x)+\omega\tau(x)\big]\mathcal{U}\Big) = \tau(x)\mathcal{U} \\
&\qquad \Longleftrightarrow \ \left\langle D_{-\omega}\Big(\big[\sigma(x)+\omega\tau(x)\big]\mathcal{U}\Big) - \tau(x)\mathcal{U}, p(x)\right\rangle = 0,\ \forall p\in\mathbb{P}, \\
&D_{q^{-1}}\Big(\big[q\sigma(x)+(q-1)x\tau(x)\big]\mathcal{U}\Big) = \tau(x)\mathcal{U} \\
&\qquad \Longleftrightarrow \quad \left\langle D_{q^{-1}}\Big(\big[q\sigma(x)+(q-1)x\tau(x)\big]\mathcal{U}\Big) - \tau(x)\mathcal{U}, p(x)\right\rangle = 0,\ \forall p\in\mathbb{P},
\end{aligned}$$

then the equivalences follow.

Finally, let us suppose that $\mathcal{U}$ is a D_ν-semiclassical linear functional of class s given by

$$D_\nu\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U} \quad \text{with} \quad s = \max\{\deg(\sigma(x))-2, \deg(\tau(x))-1\}.$$

Hence,

- if $\deg\big(\sigma(x)\big) \le s+2$ and $\deg\big(\tau(x)\big) = s+1$, then $\deg\big(\sigma(x)+\omega\tau(x)\big) \le s+2$ and $\deg\big(\tau(x)\big) = s+1$,
- if $\deg\big(\sigma(x)\big) = s+2$ and $\deg\big(\tau(x)\big) < s+1$, then $\deg\big(\sigma(x)+\omega\tau(x)\big) = s+2$ and $\deg\big(\tau(x)\big) < s+1$,

and

- if $\deg\big(\sigma(x)\big) \le s+2$ and $\deg\big(\tau(x)\big) = s+1$, then $\deg\big(q\sigma(x)+(q-1)x\tau(x)\big) \le s+2$ and $\deg\big(\tau(x)\big) = s+1$,
- if $\deg\big(\sigma(x)\big) = s+2$ and $\deg\big(\tau(x)\big) < s+1$, then $\deg\big(q\sigma(x)+(q-1)x\tau(x)\big) = s+2$ and $\deg\big(\tau(x)\big) < s+1$.

Besides, since

$$\begin{gathered}
\sigma(x) = q^{-1}\widetilde{\sigma}(x) + (q^{-1}-1)x\tau(x) \quad \text{with} \quad \widetilde{\sigma}(x) = q\sigma(x) + (q-1)x\tau(x), \\
\sigma(x) = \widehat{\sigma}(x) - \omega\tau(x) \quad \text{with} \quad \widehat{\sigma}(x) = \sigma(x) + \omega\tau(x),
\end{gathered}$$

the proof is complete. $\square$

Proposition 1.4.8. *Let $\mathcal{U}$ and $\mathcal{V}$ be two weakly quasi-definite linear functionals. If there exist nonzero polynomials $p(x)$ and $r(x)$ such that*

$$p(x)\mathcal{U} = r(x)\mathcal{V},$$

holds, i.e., $\mathcal{U}$ and $\mathcal{V}$ are related by an expression of rational type, then $\mathcal{U}$ is D_ν-semiclassical if and only if so is $\mathcal{V}$. Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s + \deg(p(x)) + \deg(r(x))$.

Proof. Let us suppose that $\mathcal{U}$ is a D_ν-semiclassical linear functional given by (1.4.3). Then $\mathcal{V}$ satisfies

$$\begin{aligned} D_\nu\big[p(x\star\nu^*)r(x)\sigma(x)\mathcal{V}\big] &= D_\nu\big[p(x\star\nu^*)p(x)\sigma(x)\mathcal{U}\big] \\ &= p(x)p(x\star\nu)D_\nu\big[\sigma(x)\mathcal{U}\big] + \hbar_\nu\Big(p(x)D_\nu[p(x)] + p(x)D_\nu[p(x\star\nu^*)]\Big)\sigma(x)\mathcal{U} \\ &= \Big(p(x\star\nu)\tau(x) + \hbar_\nu D_\nu\big[p(x)+p(x\star\nu^*)\big]\sigma(x)\Big)r(x)\mathcal{V}. \end{aligned}$$

Therefore, $\mathcal{V}$ is D_ν-semiclassical. The proof of the class follows from an easy computation. $\square$

Proposition 1.4.9 ([1, 86, 88])**.** *Let $\{P_n(x)\}_{n\geq 0}$ be a SMOP with respect to a regular linear functional $\mathcal{U}$ and let $\sigma(x)$ be a monic polynomial. $\mathcal{U}$ satisfies (1.4.3) if and only if there exists an integer $s \geq 0$ such that*

$$\sigma(x)P_n^{[1,\nu^*]}(x) = \sum_{j=n-s}^{n+\deg(\sigma(x))} \lambda_{j,n}P_j(x),\ n\geq s, \quad \text{and} \quad \lambda_{n-s,n}\neq 0,\ n\geq s+1.$$

For other characterizations of D_ν-semiclassical linear functionals see for example [1, 35, 36, 45, 81, 86, 89, 106, 108] for $\nu=\omega$, [57, 86, 95, 100] for $\nu = q$, and the references therein.

1.5 Classical Linear Functionals

A semiclassical linear functional $\mathcal{U}$ of class $s=0$, i.e., a regular linear functional $\mathcal{U}$ satisfying

$$D\big[\sigma(x)\mathcal{U}\big] = \tau(x)\mathcal{U} \quad \text{with} \quad \deg(\sigma(x))\leq 2 \quad \text{and} \quad \deg(\tau(x))=1, \tag{1.5.1}$$

is said to be *classical* and its associated SMOP is called *classical SMOP*.

For properties and characterizations of the classical SMOP as well as its classification, see for example [31, 70, 112]. One of these is as follows

Theorem 1.5.1 ([42, 43]). *Let $\mathcal{U}$ be a regular linear functional and let $\{P_n(x)\}_{n\geq 0}$ be its corresponding SMOP. The following statements are equivalent*

- **i.** *$\{P_n(x)\}_{n\geq 0}$ is a classical SMOP.*
- **ii.** *For fixed $m \geq 1$, the sequence of monic polynomials $\{P_n^{[m]}(x)\}_{n\geq 0}$ is orthogonal with respect to some (regular) linear functional $\mathcal{U}^{[m]}$.*

Moreover, if $\mathcal{U}$ satisfies (1.5.1), then

$$\mathcal{U}^{[m]} = \sigma^m(x)\mathcal{U},$$

and $\{P_n^{[m]}(x)\}_{n\geq 0}$ is also a classical SMOP of the same type as $\{P_n(x)\}_{n\geq 0}$. Indeed, $\mathcal{U}^{[m]}$ satisfies the following Pearson equation

$$D\Big[\sigma(x)\mathcal{U}^{[m]}\Big] = \big[\tau(x) + m\sigma'(x)\big]\mathcal{U}^{[m]}.$$

Under linear transformations of the variable and some conditions on the parameters, the Hermite, Laguerre and Jacobi SMOP are the only classical SMOP with respect to definite positive linear functionals. In Table 1.5.1 we describe the polynomials $\sigma(x)$ and $\tau(x)$ satisfying the Pearson equation (1.5.1), the weight function $w(x)$ (positive, integrable and supported on a infinite set of the real line) such that the classical linear functional can be represented as

$$\langle \mathcal{U}, p(x)\rangle = \int_a^b p(x)w(x)dx, \quad p \in \mathbb{P},$$

the coefficients α_n^P and β_n^P, $n \geq 1$, appearing in the TTRR (1.3.1), and the monic orthogonal polynomials $P_n^{[m]}(x)$, $n, m \geq 0$.

1.5.1 D_ν-Classical Linear Functionals

A linear functional $\mathcal{U}$ is said to be *D_ν-classical* if it is D_ν-semiclassical of class $s = 0$, i.e. it is weakly quasi-definite and it satisfies

$$D_\nu\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U}, \quad \text{with } \deg(\sigma(x)) \leq 2, \ \deg(\tau(x)) = 1. \tag{1.5.2}$$

In this case, its corresponding family of MOP is said to be *D_ν-classical*.

Remark 1.5.2. As a consequence of Proposition 1.4.7, it follows that a linear functional $\mathcal{U}$ is D_ν-classical if and only if it is D_{ν^*}-classical.

For some characterizations of the D_ν-classical families of MOP and its classification see, for instance, [1, 3, 14, 37, 64, 68, 101, 108] for $\nu = \omega$, [3, 10, 11, 14, 19, 44, 49, 58, 62, 63, 64, 65, 72, 95] for $\nu = q$, and the references therein.

One of these characterizations is related with its D_ν-derivative as follows

Table 1.5.1: Classical MOP: Hermite, Laguerre and Jacobi.

	Hermite	**Laguerre**	**Jacobi**
$P_n(x)$	$H_n(x)$	$L_n^{(\alpha)}(x)$	$P_n^{(\alpha,\beta)}(x)$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$-x+\alpha+1$	$-(\alpha+\beta+2)x+\beta-\alpha$
$(a,b)\subset\mathbb{R}$	$(-\infty,\infty)$	$(0,\infty)$	$(-1,1)$
$w(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1+x)^\beta$
Restriction	$-$	$\alpha>-1$	$\alpha>-1,\beta>-1$
α_{n+1}^P	0	$2n+\alpha+1$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$
β_{n+1}^P	$\frac{1}{2}n$	$n(n+\alpha)$	$\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}$
$P_n^{[m]}(x)$	$H_n(x)$	$L_n^{(\alpha+m)}(x)$	$P_n^{(\alpha+m,\beta+m)}(x)$

Theorem 1.5.3 ([1, 44, 58]). *A family of MOP $\{P_n(x)\}_{n=0}^{\Upsilon}$ is D_ν-classical if and only if for $1\le m<\Upsilon$ fixed, $\{P_n^{[m,\nu]}(x)\}_{n=0}^{\Upsilon-m}$ is a family of MOP.*

Moreover, if $\mathcal{U}$ is a D_ν-classical linear functional satisfying (1.5.2) and $\{P_n(x)\}_{n=0}^{\Upsilon}$ is its corresponding D_ν-classical family of MOP, then for $1\le m<\Upsilon$, $\{P_n^{[m,\nu^*]}(x)\}_{n=0}^{\Upsilon-m}$ is orthogonal with respect to the linear functional $\mathcal{U}^{[m,\nu^*]}$ given by

$$\mathcal{U}^{[m,\nu^*]}=\left(\prod_{j=0}^{m-1}\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{j\ times})\right)\mathcal{U},$$

and $\mathcal{U}^{[m,\nu^*]}$ satisfies

$$D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\ times})\mathcal{U}^{[m,\nu^*]}\right]=\left(\tau(x)+\hbar_\nu\sum_{j=1}^{m}D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{j\ times})\right]\right)\mathcal{U}^{[m,\nu^*]}.$$

Therefore, $\{P_n^{[m,\nu^*]}(x)\}_{n=0}^{\Upsilon-m}$ *is also a* D_ν*-classical family of MOP of the same type as* $\{P_n(x)\}_{n=0}^{\Upsilon}$.

Proof. We will prove the second statement (for the converse see the above references). The proof is done by induction on m. For $m=1$ we get

$$\begin{aligned}\left\langle \sigma(x)\mathcal{U}, P_\ell(x)P_n^{[1,\nu^*]}(x)\right\rangle &= \frac{1}{\eta_{n,1,\nu^*}}\left\langle D_\nu\big[\sigma(x)P_\ell(x)\mathcal{U}\big], P_{n+1}(x)\right\rangle\\ &= \frac{1}{\eta_{n,1,\nu^*}}\left\langle P_\ell(x\star\nu)D_\nu\big[\sigma(x)\mathcal{U}\big] + \hbar_\nu\left(D_\nu P_\ell\right)(x)\sigma(x)\mathcal{U}, P_{n+1}(x)\right\rangle\\ &= \frac{1}{\eta_{n,1,\nu^*}}\left\langle \mathcal{U}, \Big[P_\ell(x\star\nu)\tau(x) + \hbar_\nu\left(D_\nu P_\ell\right)(x)\sigma(x)\Big]P_{n+1}(x)\right\rangle.\end{aligned}$$

Hence,

$$\begin{gathered}\left\langle \sigma(x)\mathcal{U}, P_\ell(x)P_n^{[1,\nu^*]}(x)\right\rangle = 0, \quad 0\le \ell\le n-1,\ 1\le n\le \Upsilon-1,\\ \left\langle \sigma(x)\mathcal{U}, \left(P_n^{[1,\nu^*]}(x)\right)^2\right\rangle \neq 0, \quad 0\le n\le \Upsilon-1.\end{gathered}$$

Thus, $\{P_n^{[1,\nu^*]}(x)\}_{n=0}^{\Upsilon-1}$ is orthogonal with respect to $\mathcal{U}^{[1,\nu^*]} = \sigma(x)\mathcal{U}$. Besides, $\mathcal{U}^{[1,\nu^*]}$ satisfies

$$\begin{aligned}D_\nu\left[\sigma(x\star\nu^*)\mathcal{U}^{[1,\nu^*]}\right] &= \sigma(x)D_\nu\left[\sigma(x)\mathcal{U}\right] + \hbar_\nu D_\nu\left[\sigma(x\star\nu^*)\right]\mathcal{U}^{[1,\nu^*]}\\ &= \Big(\tau(x) + \hbar_\nu D_\nu\left[\sigma(x\star\nu^*)\right]\Big)\mathcal{U}^{[1,\nu^*]}.\end{aligned}$$

Now, we assume that the statement holds for m and then we will prove it for $m+1$ as follows

$$\begin{aligned}&\left\langle \sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}, P_\ell(x)P_n^{[m+1,\nu^*]}(x)\right\rangle\\ &\quad= \frac{\eta_{n+1,m,\nu^*}}{\eta_{n,m+1,\nu^*}}\left\langle D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})P_\ell(x)\mathcal{U}^{[m,\nu^*]}\right], P_{n+1}^{[m,\nu^*]}(x)\right\rangle\\ &\quad= \frac{\eta_{n+1,m,\nu^*}}{\eta_{n,m+1,\nu^*}}\left\langle P_\ell(x\star\nu)D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}\right]\right.\\ &\qquad\qquad\left. + \hbar_\nu\left(D_\nu P_\ell\right)(x)\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}, P_{n+1}^{[m,\nu^*]}(x)\right\rangle\\ &= \frac{\eta_{n+1,m,\nu^*}}{\eta_{n,m+1,\nu^*}}\left\langle \mathcal{U}^{[m,\nu^*]}, \left[P_\ell(x\star\nu)\left(\tau(x) + \hbar_\nu\sum_{j=1}^{m} D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{j\text{ times}})\right]\right)\right.\right.\end{aligned}$$

$$+\hbar_\nu \left(D_\nu P_\ell\right)(x)\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\Bigg] P_{n+1}^{[m,\nu^*]}(x)\Bigg\rangle.$$

As a consequence,

$$\Bigg\langle \sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}, P_\ell(x)P_n^{[m+1,\nu^*]}(x)\Bigg\rangle = 0, \quad 0\le \ell\le n-1,\ 1\le n\le \Upsilon-m-1,$$

$$\Bigg\langle \sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}, \left(P_n^{[m+1,\nu^*]}(x)\right)^2\Bigg\rangle \neq 0, \quad 0\le n\le \Upsilon-m-1,$$

i.e., $\{P_n^{[m+1,\nu^*]}(x)\}_{n=0}^{\Upsilon-m-1}$ is orthogonal with respect to $\mathcal{U}^{[m+1,\nu^*]} = \sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}$.

Furthermore,

$$\begin{aligned}
&D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m+1\text{ times}})\mathcal{U}^{[m+1,\nu^*]}\right]\\
&= \sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m\text{ times}})\mathcal{U}^{[m,\nu^*]}\right] + \hbar_\nu D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m+1\text{ times}})\right]\mathcal{U}^{[m+1,\nu^*]}\\
&\qquad = \left(\tau(x) + \hbar_\nu\sum_{j=1}^{m+1} D_\nu\left[\sigma(x\underbrace{\star\nu^*\cdots\star\nu^*}_{j\text{ times}})\right]\right)\mathcal{U}^{[m+1,\nu^*]}.
\end{aligned}$$

□

1.5.1.1 D_ω-Classical Linear Functionals

When $\omega = 1$, Kravchuk, Hahn, Charlier, and Meixner are the D_1-classical families of MOP ([37]). The linear functionals associated with Kravchuk and Hahn family of MOP are weakly quasi-definite because they have a finite set as support and their families of MOP satisfy a finite orthogonality relation. However, Charlier and Meixner linear functionals are regular ([31]). In Table 1.5.2 and Table 1.5.3, we give the polynomials $\sigma(x)$ and $\tau(x)$ which appear in (1.5.2), the weight function $w(x)$ such that the D_1-classical functional can be represented as

$$\langle \mathcal{U}, p(x)\rangle = \sum_{x_k=a}^{b-1} p(x_k)w(x_k), \quad x_{k+1} = x_k+1, \quad 0\le a,b\le\infty, \quad p\in\mathbb{P},$$

the coefficients α_n^P and β_n^P of the TTRR (1.3.1) or (1.3.8), and the MOP $P_n^{[m,1]}(x)$, $n,m\ge 0$.

Table 1.5.2: D_1-Classical MOP (Finite Support): Kravchuk and Hahn.

	Kravchuk	**Hahn**
$P_n(x)$	$K_n^{(p)}(x;N)$	$H_n^{(\alpha,\beta)}(x;N)$
$\sigma(x)$	$N-x$	$(N-x-1)(x+\beta+1)$
$\tau(x)$	$\frac{Np-x}{p}$	$(N-1)(\beta+1)-x(\alpha+\beta+2)$
x	$\{0,1,\cdots,N\}$	$\{0,1,\cdots,N-1\}$
$w(x)$	$\binom{N}{x}p^x(1-p)^{N-x}$	$\frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+x+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(x+1)}$
Restriction	$p\in(0,1), N\in\mathbb{Z}^+$	$\alpha,\beta>-1, N\in\mathbb{Z}^+$
α_{n+1}^P	$n+p(N-2n)$	$\frac{\alpha-\beta+2N-2}{4}+\frac{(\beta^2-\alpha^2)(\alpha+\beta+2N)}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$
β_{n+1}^P	$pn(1-p)(N-n+1)$	$\frac{n(N-n)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$
$P_n^{[m,1]}(x)$	$K_n^{(p)}(x;N-m)$	$H_n^{(\alpha+m,\beta+m)}(x;N-m)$

1.5.1.2 D_q-Classical Linear Functionals

Big q-Jacobi, little q-Jacobi, little q-Laguerre/Wall, big q-Laguerre, q-Meixner, q-Bessel or alternative q-Charlier, q-Laguerre or Moak, Al-Salam-Carlitz I, Al-Salam-Carlitz II, Stieltjes-Wigert, discrete q-Hermite I, and discrete q-Hermite II are families of q-classical orthogonal polynomials with respect to positive definite linear functionals (except for either q changes by q^{-1}, or parameter changes) ([44, 65]). These families of orthogonal polynomials are those belonging to the q-Hahn tableau [44, p. 113]. In Table 1.5.4, Table 1.5.5, and Table 1.5.6, we state the polynomials $\sigma(x)$ and $\tau(x)$ of the D_q-Pearson equation (1.5.2), as well as the coefficients α_n^P and β_n^P of the TTRR (1.3.1). In these tables, we also give the weight function $w(x)$ (which includes factors such as $(\alpha;q)_\infty$ which converge whenever $|q|<1$) such that the D_q-classical functional can be represented as

$$\langle\mathcal{U},p(x)\rangle=\sum_{k=-\infty}^{\infty}p(\zeta q^k)w(k),\quad\text{or}\quad\langle\mathcal{U},p(x)\rangle=\int p(x)w(x)d_qx,\quad p\in\mathbb{P},$$

Table 1.5.3: D_1-Classical MOP (Infinite Support): Charlier and Meixner.

	Charlier	**Meixner**
$P_n(x)$	$C_n^{(\mu)}(x)$	$M_n^{(\gamma,\mu)}(x)$
$\sigma(x)$	μ	$\mu(\gamma+x)$
$\tau(x)$	$\mu - x$	$\mu\gamma - x(1-\mu)$
x	$\{0,1,2,\cdots\}$	$\{0,1,2,\cdots\}$
$w(x)$	$\frac{e^{-\mu}\mu^x}{\Gamma(x+1)}$	$\frac{\mu^x(1-\mu)^\gamma\Gamma(x+\gamma)}{\Gamma(x+1)\Gamma(\gamma)}$
Restriction	$\mu > 0$	$\gamma > 0, \mu \in (0,1)$
α_{n+1}^P	$n+\mu$	$\frac{\gamma\mu+n(1+\mu)}{1-\mu}$
β_{n+1}^P	$n\mu$	$\frac{\mu n(\gamma+n-1)}{(1-\mu)^2}$
$P_n^{[m,1]}(x)$	$C_n^{(\mu)}(x)$	$M_n^{(\gamma+m,\mu)}(x)$

where ζ is a constant and $\int \cdot\, d_qx$ is the *Jackson integral* which defined by ([51, 56, 62])

$$\int p(x)d_qx = (1-q)x\sum_{k=0}^{\infty} q^k p(q^k x).$$

On the other hand, the mth D_q-derivatives (which are also MOP) of the Big q-Jacobi, little q-Jacobi and little q-Laguerre/Wall MOP are, respectively,

$$P_n^{[m,q]}(x;q^\alpha,q^\beta,-q^\gamma;q) = q^{-mn}P_n(q^m x;q^{\alpha+m},q^{\beta+m},-q^{\gamma+m};q), \quad n,m\geq 0,$$
$$p_n^{[m,q]}(x;q^\alpha,q^\beta|q) = p_n(x;q^{\alpha+m},q^{\beta+m}|q), \quad n,m\geq 0,$$
$$p_n^{[m,q]}(x;q^\alpha|q) = p_n(x;q^{\alpha+m}|q), \quad n,m\geq 0.$$

Table 1.5.4: q-Classical MOP: Big q-Jacobi, Little q-Jacobi, and Little q-Laguerre/Wall.

	Big q-Jacobi
$P_n(x)$	$P_n(x;a,b,c;q)$
Restriction	$0<a<q^{-1}, 0\leq b<q^{-1}, c<0$
$\sigma(x)$	$aq^2(x-1)(bx-c)$
$\tau(x)$	$\frac{cq-x+aq(1-(b+c)q+bqx)}{q-1}$
α_{n+1}^P	$\frac{q^{1+n}(c+a(1+b+c+b(c+a(1+b+c))q^{1+2n}-(c+b(1+a+c))q^n(1+q)))}{(abq^{2n}-1)(abq^{2+2n}-1)}$
β_{n+1}^P	$-\frac{aq^{2+n}(q^n-1)(aq^n-1)(bq^n-1)(abq^n-1)(abq^n-c)(cq^n-1)}{(abq^{2n}-1)^2(abq^{2n}-q)(abq^{1+2n}-1)}$
$w(x)$	$\frac{(a^{-1}x;q)_\infty(c^{-1}x;q)_\infty}{(x;q)_\infty(bc^{-1}x;q)_\infty}$
$\langle\mathcal{U},p(x)\rangle$	$\int_{cq}^{aq}p(x)w(x)d_qx$

	Little q-Jacobi	**Little q-Laguerre/Wall**
$P_n(x)$	$p_n(x;a,b\|q)$	$p_n(x;a\|q)$
Restriction	$0<a<q^{-1}, b<q^{-1}$	$0<a<q^{-1}$
$\sigma(x)$	$aqx(bqx-1)$	aqx
$\tau(x)$	$\frac{1-x+aq(bqx-1)}{q-1}$	$\frac{-1+aq+x}{q-1}$
α_{n+1}^P	$\frac{q^n(1+a+a(1+a)bq^{1+2n}-a(1+b)q^n(1+q))}{(abq^{2n}-1)(abq^{2+2n}-1)}$	$q^n(1+a-aq^n(1+q))$
β_{n+1}^P	$\frac{aq^{2n}(q^n-1)(aq^n-1)(bq^n-1)(abq^n-1)}{(abq^{2n}-1)^2(abq^{2n}-q)(abq^{1+2n}-1)}$	$aq^{-1+2n}(q^n-1)(aq^n-1)$
$w(x)$	$\frac{(bq;q)_k}{(q;q)_k}(aq)^k$	$\frac{1}{(q;q)_k}(aq)^k$
$\langle\mathcal{U},p(x)\rangle$	$\sum_{k=0}^\infty p(q^k)w(k)$	$\sum_{k=0}^\infty p(q^k)w(k)$

Table 1.5.5: q-Classical MOP: Big q-Laguerre, q-Meixner, q-Bessel or Alternative q-Charlier, and q-Laguerre or Moak.

	Big q-Laguerre	**q-Meixner**
$P_n(x)$	$P_n(x;a,b;q)$	$M_n(x;b,c;q)$
Restriction	$0<a<q^{-1}, b<0$	$0\le b<q^{-1}, c>0$
$\sigma(x)$	$abq^2(x-1)$	$q(x-1)(x+bc)$
$\tau(x)$	$\frac{-(a+b)q+abq^2+x}{q-1}$	$\frac{c(bq-1)+q(x-1)}{q-1}$
α_{n+1}^P	$q^n((a+b+ab)q-abq^{1+n}(1+q))$	$\frac{c(1+q^{-1})-(-1+c+bc)q^n}{q^{2n}}$
β_{n+1}^P	$abq^{1+n}(q^n-1)(aq^n-1)(bq^n-1)$	$\frac{c(q^n-1)(c+q^n)(bq^n-1)}{q^{4n-1}}$
$w(x)$	$\frac{(a^{-1}x;q)_\infty(b^{-1}x;q)_\infty}{(x;q)_\infty}$	$\frac{(bq;q)_k}{(q;q)_k(-bcq;q)_k}\, c^k q^{k(k-1)/2}$
$\langle\mathcal{U},p(x)\rangle$	$\int_{bq}^{aq} p(x)w(x)d_qx$	$\sum_{k=0}^\infty p(q^{-k})w(k)$

	q-Bessel or Alternative q-Charlier	**q-Laguerre or Moak**
$P_n(x)$	$C_n(x;a;q)$	$L_n^{(\alpha)}(x;q)$
Restriction	$a>0$	$\alpha>-1$
$\sigma(x)$	aqx^2	$q^{1+\alpha}x((q-1)x-1)$
$\tau(x)$	$\frac{-1+x(1+aq)}{q-1}$	$\frac{1+q^{1+\alpha}(x(q-1)-1)}{q-1}$
α_{n+1}^P	$\frac{q^n(q-aq^{1+2n}+aq^n(1+q))}{(q+aq^{2n})(1+aq^{1+2n})}$	$\frac{q^{-1-\alpha-2n}(-1+q(-1+q^n+q^{\alpha+n}))}{q-1}$
β_{n+1}^P	$-\frac{aq^{1+3n}(q^n-1)(q+aq^n)}{(1+aq^{2n})(q+aq^{2n})^2(q^2+aq^{2n})}$	$q^{-1-2\alpha-4n}[n]_q[\alpha+n]_q$
$w(x)$	$\frac{1}{(q;q)_k}\, a^k q^{(k+1)k/2}$	$\frac{1}{(-cq^k;q)_\infty}\, q^{k(\alpha+1)}$, $c>0$
$\langle\mathcal{U},p(x)\rangle$	$\sum_{k=0}^\infty p(q^k)w(k)$	$\sum_{k=-\infty}^\infty p(cq^k)w(k)$, $c>0$

Table 1.5.6: q-Classical MOP: Al-Salam-Carlitz I, Al-Salam-Carlitz II, Stieltjes-Wigert, Discrete q-Hermite I, and Discrete q-Hermite II.

	Al-Salam-Carlitz I	**Al-Salam-Carlitz II**	**Stieltjes-Wigert**
$P_n(x)$	$U_n^{(a)}(x;q)$	$V_n^{(a)}(x;q)$	$S_n(x;q)$
Restriction	$a<0$	$0<a<q^{-1}$	-
$\sigma(x)$	a	$(x-a)(x-1)$	qx^2
$\tau(x)$	$\frac{1+a-x}{q-1}$	$\frac{x-a-1}{q-1}$	$\frac{qx-1}{q-1}$
α_{n+1}^P	$(1+a)q^n$	$(1+a)q^{-n}$	$q^{-2n}(1+q^{-1}-q^n)$
β_{n+1}^P	$aq^{n-1}(q^n-1)$	$-aq^{1-2n}(q^n-1)$	$-q^{1-4n}(q^n-1)$
$w(x)$	$(qx;q)_\infty(a^{-1}qx;q)_\infty$	$\frac{1}{(q;q)_k(aq;q)_k}\,a^kq^{k^2}$	$\frac{1}{(-x;q)_\infty(-qx^{-1};q)_\infty}$
$\langle\mathcal{U},p(x)\rangle$	$\int_a^1 p(x)w(x)d_qx$	$\sum_{k=0}^\infty p(q^{-k})w(k)$	$\int_0^\infty p(x)w(x)d_qx$

	Discrete q-Hermite I	**Discrete q-Hermite II**
$P_n(x)$	$h_n(x;q)$	$\widetilde{h}_n(x;q)$
Restriction	-	-
$\sigma(x)$	1	x^2-1
$\tau(x)$	$\frac{x}{q-1}$	$\frac{x}{q-1}$
α_{n+1}^P	0	0
β_{n+1}^P	$-q^{n-1}(q^n-1)$	$q^{1-2n}(q^n-1)$
$w(x)$	$(qx;q)_\infty(-qx;q)_\infty$	$\frac{1}{(ix;q)_\infty(-ix;q)_\infty}\,q^k=\frac{1}{(-x^2;q^2)_\infty}\,q^k$
$\langle\mathcal{U},p(x)\rangle$	$\int_{-1}^1 p(x)w(x)d_qx$	$\sum_{k=-\infty}^\infty[p(cq^k)+p(-cq^k)]w(k)$, $c>0$

1.6 Orthogonal Polynomial on the Unit Circle

1.6.1 Linear Functionals

Let us consider the unit circle

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

the linear space of Laurent polynomials with complex coefficients

$$\Lambda = \operatorname{span}_{\mathbb{C}}\{z^n : n \in \mathbb{Z}\},$$

and a linear functional $\mathcal{U} : \Lambda \longrightarrow \mathbb{C}$. We can associate with $\mathcal{U}$ a *sequence of moments* $\{c_n\}_{n\in\mathbb{Z}}$ defined by

$$c_n = \langle \mathcal{U}, z^n \rangle, \quad n \in \mathbb{Z},$$

and a bilinear form as follows

$$\big\langle p(z), q(z) \big\rangle_{\mathcal{U}} = \big\langle \mathcal{U}, p(z)\overline{q}(1/z) \big\rangle,$$

where $p, q \in \mathbb{P}$, the linear space of polynomials with complex coefficients, and $\big\langle \mathcal{U}, f(z) \big\rangle$ denotes the image of the Laurent polynomial $f(z)$ by the linear functional $\mathcal{U}$. Notice that

$$\big\langle p(z), q(z) \big\rangle_{\mathcal{U}} = \big\langle p(z)\overline{q}(1/z), 1 \big\rangle_{\mathcal{U}} = \big\langle 1, \overline{p}(1/z)q(z) \big\rangle_{\mathcal{U}} = \big\langle \overline{q}(1/z), \overline{p}(1/z) \big\rangle_{\mathcal{U}}, \quad p, q \in \mathbb{P}.$$

The Gram matrix of the linear functional $\mathcal{U}$ with respect to the canonical basis $\{z^n\}_{n\geq 0}$ is the infinite Toeplitz matrix T

$$T = [c_{j-k}]_{k,j\geq 0} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots \\ c_{-1} & c_0 & c_1 & \ddots \\ c_{-2} & c_{-1} & c_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad c_j = \big\langle \mathcal{U}, z^j \big\rangle.$$

T_n, $n \geq 0$, will denote their leading principal submatrices

$$T_n = \big[c_{j-k}\big]_{k,j=0}^{n}, \quad n \geq 0.$$

When

$$c_{-n} = \overline{c}_n, \quad n \geq 0,$$

the linear functional $\mathcal{U}$ is called *Hermitian* linear functional.

On the other hand, an analytic function $F(z)$, defined on the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

is said to be a *Carathéodory function* if and only if ([111])

$$F(0) = 1 \quad \text{and} \quad \mathrm{Re}F(z) > 0, \quad z \in \mathbb{D}.$$

If μ is a nontrivial probability measure on $\mathbb{T}$, then the function defined by

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta) \tag{1.6.1}$$

is a Carathéodory function. Conversely, the *Herglotz representation theorem* claims that every Carathéodory function $F(z)$ has an integral representation as (1.6.1) for a unique nontrivial probability measure μ on $\mathbb{T}$ ([111]).

Besides, a Carathéodory function (1.6.1), has the expansions

$$F(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n}z^n, \quad |z| < 1,$$
$$F(z) = -c_0 - 2\sum_{n=1}^{\infty} c_n z^{-n}, \quad |z| > 1,$$

where $\{c_n\}_{n\geq 0}$ is the sequence of the moments of the measure associated with $F(z)$ ([104]).

If $\mathcal{U}$ is a Hermitian linear functional and $\mathcal{U}_\theta$ means that $\mathcal{U}$ operates on the variable $e^{i\theta}$, then we get

$$\frac{1}{2\pi}\left\langle \mathcal{U}_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\right\rangle = c_0 + 2\sum_{n=1}^{\infty} c_{-n}z^n, \quad |z| < 1,$$
$$\frac{1}{2\pi}\left\langle \mathcal{U}_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\right\rangle = -c_0 - 2\sum_{n=1}^{\infty} c_n z^{-n}, \quad |z| > 1,$$

and thus, the function defined by

$$F(z) = \frac{1}{2\pi}\left\langle \mathcal{U}_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\right\rangle, \quad |z| \neq 1,$$

is called the *formal Carathéodory function* associated with $\mathcal{U}$.

On the other hand, as in the case of polynomials of one variable on the real line, for a fixed $m \geq 0$, $\phi_n^{[m]}(z)$ will denote the monic mth derivative of the monic polynomial of degree $n+m$, $\phi_{n+m}(z)$, $n \geq 0$, which is the monic polynomial of degree n

$$\phi_n^{[m]}(z) = \frac{\phi_{n+m}^{(m)}(z)}{(n+1)_m}, \quad m, n \geq 0,$$

where $(n+1)_m$ denotes the Pochhammer symbol defined in (1.1.8).

1.6.2 Orthogonal Polynomials on the Unit Circle

An Hermitian linear functional $\mathcal{U}$ is said to be *quasi-definite or regular* if all leading principal submatrices T_n of its associated Toeplitz matrix are nonsingular, i.e., $\det(T_n) \neq 0$ for $n \geq 0$, and it is called *positive definite* if all these leading principal submatrices are positive definite, i.e., $\det(T_n) > 0$ for $n \geq 0$.

In this way, an Hermitian linear functional $\mathcal{U}$ is regular if and only if there exists a (unique) *sequence of monic orthogonal polynomials on the unit circle* (OPUC), i.e., if there exists a sequence of monic polynomials $\{\phi_n(z)\}_{n\geq 0}$ such that

- $\deg(\phi_n(z)) = n$, $n \geq 0$, and
- $\langle \phi_m(z), \phi_n(z) \rangle_{\mathcal{U}} = \langle \mathcal{U}, \phi_m(z)\overline{\phi}_n(1/z) \rangle = \kappa_n \delta_{m,n}$, with $\kappa_n \neq 0$.

In this case, every polynomial $p_n(z)$ of degree n, $n \geq 0$, can be expanded in terms of the sequence of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ as

$$p_n(z) = \sum_{j=0}^{n} \zeta_{j,n} \phi_j(z),$$

where

$$\zeta_{j,n} = \frac{\langle p_n(z), \phi_j(z) \rangle_{\mathcal{U}}}{\langle \phi_j(z), \phi_j(z) \rangle_{\mathcal{U}}} = \frac{\langle \mathcal{U}, p_n(z)\overline{\phi}_j(1/z) \rangle}{\langle \mathcal{U}, \phi_j(z)\overline{\phi}_j(1/z) \rangle}.$$

Besides, every monic OPUC $\phi_n(z)$, $n \geq 0$, has an explicit representation, the so called *Heine's formula*, as follows

$$\phi_n(z) = \frac{1}{\det(T_{n-1})} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad n \geq 1, \quad \phi_0(z) = 1.$$

This sequence of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ satisfies the *forward Szegő recurrence relation*

$$\phi_n(z) = z\phi_{n-1}(z) + \alpha_n \phi_{n-1}^*(z), \;\; n \geq 1, \quad \phi_0(z) = 1, \tag{1.6.2}$$

and the *backward Szegő recurrence relation*

$$\phi_n(z) = \left(1 - |\alpha_n|^2\right) z\phi_{n-1}(z) + \alpha_n \phi_n^*(z), \;\; n \geq 1, \quad \phi_0(z) = 1,$$

where $\phi_n^*(z)$ denotes the *reversed polynomial* of $\phi_n(z)$ defined by

$$\phi_n^*(z) = z^n \overline{\phi}_n(1/z), \quad n \geq 0,$$

and

$$\alpha_n = \phi_n(0), \quad n \geq 0.$$

$\{\alpha_n\}_{n\geq 0}$ is said to be the sequence of *Verblunsky, reflection, Schur, Szegő or Geronimus coefficients* of the regular Hermitian linear functional $\mathcal{U}$ ([39, 40, 41, 111, 112]).

Remark 1.6.1. If $\mathcal{U}$ is a regular Hermitian linear functional and $\{\phi_n(z)\}_{n\geq 0}$ is its corresponding sequence of monic OPUC, then (1.6.2) holds, so applying $\left\langle \cdot, z\phi_{n-1}(z)\right\rangle_{\mathcal{U}}$ to both sides of this equation it follows that, for $n \geq 1$,

$$\begin{aligned}
\left\langle \phi_n(z), z\phi_{n-1}(z)\right\rangle_{\mathcal{U}} &= \left\langle z\phi_{n-1}(z), z\phi_{n-1}(z)\right\rangle_{\mathcal{U}} + \left\langle \alpha_n\phi_{n-1}^*(z), z\phi_{n-1}(z)\right\rangle_{\mathcal{U}} \\
&\overset{(1.6.2)}{=} \left\langle \phi_{n-1}(z), \phi_{n-1}(z)\right\rangle_{\mathcal{U}} + \alpha_n \left\langle \phi_{n-1}^*(z), \phi_n(z) - \alpha_n\phi_{n-1}^*(z)\right\rangle_{\mathcal{U}} \\
&= \left\langle \phi_{n-1}(z), \phi_{n-1}(z)\right\rangle_{\mathcal{U}} - |\alpha_n|^2 \left\langle \phi_{n-1}^*(z), \phi_{n-1}^*(z)\right\rangle_{\mathcal{U}},
\end{aligned}$$

and since

$$\langle \phi_n^*(z), \phi_n^*(z)\rangle_{\mathcal{U}} = \left\langle z^n\overline{\phi}_n(1/z), z^n\overline{\phi}_n(1/z)\right\rangle_{\mathcal{U}} = \left\langle \phi_n(z), \phi_n(z)\right\rangle_{\mathcal{U}}, \quad n \geq 0,$$

then

$$\begin{aligned}
\left\langle \phi_n(z), \phi_n(z)\right\rangle_{\mathcal{U}} &= \left(1 - |\alpha_n|^2\right) \left\langle \phi_{n-1}(z), \phi_{n-1}(z)\right\rangle_{\mathcal{U}} \\
&= \left\langle \phi_0(z), \phi_0(z)\right\rangle_{\mathcal{U}} \prod_{j=1}^{n} \left(1 - |\alpha_j|^2\right) = c_0 \prod_{j=1}^{n} \left(1 - |\alpha_j|^2\right), \quad n \geq 1.
\end{aligned}$$

Therefore, $\mathcal{U}$ is regular if and only if

$$|\alpha_n| \neq 1, \quad n \geq 1.$$

Furthermore, $\mathcal{U}$ is positive definite if and only if

$$|\alpha_n| < 1, \quad n \geq 1.$$

Additionally, the sequence of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ is completely determined by the sequence of Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$. Indeed, if $\{\phi_n(z)\}_{n\geq 0}$ is a sequence of monic polynomials defined by (1.6.2) and $\phi_n(0) = \alpha_n$, with $|\alpha_n| \neq 1$, $n \geq 1$, then $\{\phi_n(z)\}_{n\geq 0}$ is orthogonal with respect to some regular Hermitian linear functional. Moreover, every sequence of complex numbers $\{\alpha_n\}_{n\geq 1}$ such that $|\alpha_n| < 1$, $n \geq 1$, is the sequence of Verblunsky coefficients associated with some positive definite Hermitian linear functional ([109, 110, 113]). This result is the analogous of the Favard theorem on the real line (see Theorem 1.3.1).

Thus, if $\{\phi_n(z)\}_{n\geq 0}$ is a sequence of monic OPUC with Verblunsky coefficients $\{\alpha_n\}_{n\geq 1}$, then for fixed $N \geq 0$, we define the following polynomials

$$\phi_n^{(N)}(z) = z\phi_{n-1}(z) + \alpha_{n+N}\phi_{n-1}^*(z), \;\; n \geq 1, \quad \phi_0^{(N)}(z) = 1,$$

which are called the *associated polynomials of* $\{\phi_n(z)\}_{n\geq 0}$ *of order* N. Similarly, given a finite set of complex numbers $\{\gamma_n\}_{n=1}^{N}$, with $|\gamma_n| \neq 1$, $n = 1, 2, ..., N$, we can define new Verblunsky coefficients

$$\{\tilde{\alpha}_n\}_{n\geq 1} = \{\gamma_1, \dots, \gamma_N, \alpha_1, \alpha_2, \dots\}.$$

Then the monic OPUC defined by the forward Szegő relation (1.6.2) associated with $\{\tilde{\alpha}_n\}_{n\geq 1}$ are said to be the *anti-associated polynomials of* $\{\phi_n(z)\}_{n\geq 0}$ *of order* N.

The following theorem characterizes a sequence of anti-associated polynomials from its corresponding initial sequence of monic OPUC. More precisely, it shows how the orthogonal polynomials and the Carathéodory function $F(z)$ changes if the sequence of Verblunsky coefficients is extended backward or altered at a finite number of places ([103]).

Theorem 1.6.2 ([103]). *Let* $\gamma_1, \dots, \gamma_N, \alpha_1, \alpha_2, \dots \in \mathbb{C}$ *be such that* $|\gamma_n| \neq 1$ *for* $n = 1, 2, ..., N$ *and* $|\alpha_n| \neq 1$ *for* $n \geq 1$. *Furthermore, let* $\{\phi_n(z)\}_{n\geq 0}$ *and* $\{\varphi_n(z)\}_{n\geq 0}$ *be the sequences of monic OPUC generated by the coefficients* $\{\alpha_n\}_{n\geq 0}$ *and* $\{-\alpha_n\}_{n\geq 0}$ *respectively (using 1.6.2), and, let* $\widetilde{\phi}_N(z)$ *and* $\widetilde{\varphi}_N$ *be the monic polynomials generated by* $\widetilde{\phi}_0(z) = 1 = \widetilde{\varphi}_0(z)$ *and*

$$\begin{aligned}
\widetilde{\phi}_n(z) &= z\widetilde{\phi}_{n-1}(z) + \gamma_n \widetilde{\phi}^*_{n-1}(z), \quad n = 1, \dots, N,\\
\widetilde{\varphi}_n(z) &= z\widetilde{\varphi}_{n-1}(z) - \gamma_n \widetilde{\varphi}^*_{n-1}(z), \quad n = 1, \dots, N.
\end{aligned}$$

Then the polynomials $\{\widetilde{\phi}_n(z)\}_{n\geq 0}$ *and* $\{\widetilde{\varphi}_n(z)\}_{n\geq 0}$ *generated by the Verblunsky coefficients*

$$\{\gamma_1, \dots, \gamma_N, \alpha_1, \alpha_2, \dots\}$$

are given by

$$\begin{aligned}
2\,\widetilde{\phi}_{n+N}(z) &= \Big(\widetilde{\phi}_N(z) + \widetilde{\phi}^*_N(z)\Big)\phi_n(z) + \Big(\widetilde{\phi}_N(z) - \widetilde{\phi}^*_N(z)\Big)\varphi_n(z), \quad n \geq 0,\\
2\,\widetilde{\varphi}_{n+N}(z) &= \Big(\widetilde{\varphi}_N(z) + \widetilde{\varphi}^*_N(z)\Big)\varphi_n(z) + \Big(\widetilde{\varphi}_N(z) - \widetilde{\varphi}^*_N(z)\Big)\phi_n(z), \quad n \geq 0.
\end{aligned}$$

They are orthogonal with respect to a linear functional such that the corresponding Carathéodory function is

$$\widetilde{F}(z) = \frac{\Big(\widetilde{\varphi}^*_N(z) - \widetilde{\varphi}_N(z)\Big) + \Big(\widetilde{\varphi}_N(z) + \widetilde{\varphi}^*_N(z)\Big)F(z)}{\Big(\widetilde{\phi}_N(z) + \widetilde{\phi}^*_N(z)\Big) + \Big(\widetilde{\phi}^*_N(z) - \widetilde{\phi}_N(z)\Big)F(z)},$$

where $F(z)$ *is the Carathéodory function such that the polynomials* $\phi_n(z)$, $n \geq 0$, *are orthogonal with respect to the associated linear functional.*

On the other hand, for every positive definite Hermitian linear functional $\mathcal{U}$ we have an integral representation of the associated inner product as ([111])

$$\big\langle p(z), q(z)\big\rangle_{\mathcal{U}} = \big\langle \mathcal{U}, p(z)\overline{q}(1/z)\big\rangle = \frac{1}{2\pi}\int_0^{2\pi} p(z)\overline{q}(1/z)d\mu(\theta), \quad z = e^{i\theta},\ p, q \in \mathbb{P},$$

where μ is a nontrivial probability measure supported on an infinite subset of $\mathbb{T}$. This measure μ can be decomposed into a part that is purely absolutely continuous with respect to the Lebesgue measure $d\theta/2\pi$ and a singular part as follows

$$d\mu(\theta) = d\mu_{ac}(\theta) + d\mu_s(\theta) = \mu'(\theta)\frac{d\theta}{2\pi} + d\mu_s(\theta),$$

where μ' denotes the Radon-Nikodym derivative of the measure μ with respect to the Lebesgue measure $d\theta/2\pi$.

A measure μ is said to belong to the *Nevai class* ([93, 94]) if its corresponding sequence of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ satisfies

$$\lim_{n\to\infty} |\phi_n(0)| = 0.$$

Remark 1.6.3. All zeros of every monic polynomial $\phi_n(z)$, $n \geq 1$, orthogonal with respect a nontrivial probability measure μ supported on an infinite subset of $\mathbb{T}$, are in the unit disk $\mathbb{D}$. Indeed, if

$$\phi_n(z) = (z-\xi)q_{n-1}(z), \quad \deg(q_{n-1}(z)) = n-1,\ n \geq 1,$$

then

$$\begin{aligned}
0 < \|\phi_n(z)\|_\mu^2 &= \langle \phi_n(z), \phi_n(z)\rangle_\mu = \langle \phi_n(z), (z-\xi)q_{n-1}(z)\rangle_\mu = \langle \phi_n(z), zq_{n-1}(z)\rangle_\mu \\
&= \langle (z-\xi)q_{n-1}(z), zq_{n-1}(z)\rangle_\mu = \|q_{n-1}(z)\|_\mu^2 - \langle \xi q_{n-1}(z), (z-\xi+\xi)q_{n-1}(z)\rangle_\mu \\
&= \|q_{n-1}(z)\|_\mu^2 - |\xi|^2 \|q_{n-1}(z)\|_\mu^2 = \left(1-|\xi|^2\right)\|q_{n-1}(z)\|_\mu^2 .
\end{aligned}$$

Thus,

$$|\xi| < 1.$$

In this way, according to the Szegő forward recurrence relation, if μ belongs to the Nevai class, then

$$\left|\frac{\phi_n(z)}{\phi_{n-1}(z)} - z\right| \leq |\phi_n(0)|, \quad z \in \mathbb{C}\setminus\mathbb{D},$$

and, therefore,

$$\lim_{n\to\infty}\frac{\phi_n(z)}{\phi_{n-1}(z)} = z$$

uniformly in compact subsets of $\mathbb{C}\setminus\mathbb{D}$.

Now, let us consider the multiplication operator by z as follows

$$z\phi_n(z) = \phi_{n+1}(z) + \sum_{j=0}^{n} \zeta_{j,n+1}\phi_j(z), \quad n \geq 0,$$

where $\phi_n(z)$, $n \geq 0$, are the monic OPUC with respect to a regular Hermitian linear functional $\mathcal{U}$. Hence, for $0 \leq j \leq n$ and $n \geq 0$,

$$\begin{aligned}\left\langle \phi_j(z), \phi_j(z) \right\rangle_{\mathcal{U}} \zeta_{j,n+1} = \left\langle z\phi_n(z), \phi_j(z) \right\rangle_{\mathcal{U}} &\overset{(1.6.2)}{=} \left\langle \phi_{n+1}(z) - \alpha_{n+1}\phi_n^*(z), \phi_j(z) \right\rangle_{\mathcal{U}} \\ &= -\alpha_{n+1}\left\langle \phi_n^*(z), \phi_j(z) \right\rangle_{\mathcal{U}},\end{aligned}$$

with

$$\left\langle \phi_n^*(z), \phi_0(z) \right\rangle_{\mathcal{U}} = \left\langle z^n\overline{\phi}_n(1/z), 1 \right\rangle_{\mathcal{U}} = \left\langle z^n, \phi_n(z) \right\rangle_{\mathcal{U}} = \left\langle \phi_n(z), \phi_n(z) \right\rangle_{\mathcal{U}},$$

and for $1 \leq j \leq n$

$$\begin{aligned}\left\langle \phi_n^*(z), \phi_j(z) \right\rangle_{\mathcal{U}} &\underset{j\geq 1}{\overset{(1.6.2)}{=}} \left\langle \phi_n^*(z), z\phi_{j-1}(z) + \alpha_j\phi_{j-1}^*(z) \right\rangle_{\mathcal{U}} \\ &= \left\langle z^n\overline{\phi}_n(1/z), z\phi_{j-1}(z) + \alpha_j z^{j-1}\overline{\phi}_{j-1}(1/z) \right\rangle_{\mathcal{U}} \\ &= \left\langle z^n\left[z^{-1}\overline{\phi}_{j-1}(1/z) + \overline{\alpha}_j z^{1-j}\phi_{j-1}(z)\right], \phi_n(z) \right\rangle_{\mathcal{U}} \\ &= \overline{\alpha}_j\left\langle \phi_n(z), \phi_n(z) \right\rangle_{\mathcal{U}}.\end{aligned}$$

Therefore

$$z\phi_n(z) = \phi_{n+1}(z) - \alpha_{n+1}\left\langle \phi_n(z), \phi_n(z) \right\rangle_{\mathcal{U}} \sum_{j=0}^{n} \frac{\overline{\alpha}_j}{\left\langle \phi_j(z), \phi_j(z) \right\rangle_{\mathcal{U}}} \phi_j(z), \quad n \geq 0.$$

Consequently, the matrix representation of the multiplication operator by z in terms of the basis $\{\phi_n(z)\}_{n\geq 0}$ is

$$z\Phi(z) = \mathcal{H}_\phi \Phi(z), \tag{1.6.3}$$

where

$$\Phi(z) = \left[\begin{array}{ccc} \phi_0(z), & \phi_1(z), & \cdots \end{array}\right]^T$$

and $\mathcal{H}_\phi$ is the infinite lower Hessenberg matrix with entries

$$\mathcal{H}_\phi = \left[h_{k,j}\right]_{k,j\geq 0} = \begin{cases} 1, & \text{if } j = k+1, \\ -\frac{\left\langle \phi_n(z), \phi_n(z) \right\rangle_{\mathcal{U}}}{\left\langle \phi_j(z), \phi_j(z) \right\rangle_{\mathcal{U}}} \alpha_{n+1}\overline{\alpha}_j, & \text{if } j \leq k, \\ 0, & \text{if } j > k+1. \end{cases}$$

1.6.3 The Lebesgue and Bernstein-Szegő Linear Functionals

The *Lebesgue and Bernstein-Szegő linear functionals* are two well known and elementary positive definite linear functionals associated with the measures

Lebesgue measure: $\quad d\mu(\theta) = \dfrac{d\theta}{2\pi},$

Bernstein-Szegő measure with parameter $-C$: $d\mu(\theta) = \dfrac{1-|C|^2}{|1+Ce^{i\theta}|^2}\dfrac{d\theta}{2\pi}$, $C \in \mathbb{C}$, $|C| < 1$.

In the Table 1.6.1, for each of these measures we give their corresponding sequence of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$, their reversed polynomials $\phi_n^*(z)$, $n \geq 0$, their Verblunsky coefficients $\alpha_n = \phi_n(0)$, $n \geq 1$, their sequence of moments $\{c_n\}_{n\geq 0}$, and their Carathéodory function $F(z)$.

Table 1.6.1: Lebesgue and Bernstein-Szegő SMOPUC.

	Lebesgue	**Bernstein-Szegő**
Parameter	-	$-C\,:\,C \in \mathbb{C},\ \lvert C\rvert < 1$
$d\mu(\theta)$	$\frac{d\theta}{2\pi}$	$\frac{1-\lvert C\rvert^2}{\lvert 1+Ce^{i\theta}\rvert^2}\frac{d\theta}{2\pi}$
$\phi_n(z)$	z^n	$\phi_0(z)=1,\ \phi_n(z)=z^{n-1}(z+C),\ n\geq 1$
$\phi_n^*(z)$	1	$\phi_0^*(z)=1,\ \phi_n^*(z)=1+\overline{C}z,\ n\geq 1$
α_n	0	$\alpha_1=C,\ \alpha_n=0,\ n\geq 2$
c_n	$\delta_{n,0}$	$(-C)^n$
$F(z)$	1	$\frac{1-zC}{1+zC}$

CHAPTER 2

Coherent Pairs, Continuous Case

In this chapter, we will consider regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ and their associated SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, respectively, such that their mth and kth derivatives are related by

$$\sum_{i=0}^{M} a_{i,n} P_{n-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n-i}^{(k)}(x), \quad n \geq 0,$$

where M, N, m, k are non-negative integers, and $\{a_{i,n}\}_{n\geq 0}$, $\{b_{i,n}\}_{n\geq 0}$ are sequences of complex numbers such that $a_{M,n} \neq 0$, $n \geq M$, $b_{N,n} \neq 0$, $n \geq N$, and $a_{i,n} = b_{i,n} = 0$, $i > n$. In this case, $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-*coherent pair of order* (m, k). Also, when $k = 0$, $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-*coherent pair of order* m, and when $(m, k) = (1, 0)$, it is called a (standard) (M, N)-*coherent pair* (see [48]).

The structure of this chapter is as follows. In Section 2.1, we will focus our attention on the algebraic properties of a (M, N)-coherent pair of order (m, k). To be more precise, we will show that if $m = k$, then $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor and, if $m \neq k$, then $\mathcal{U}$ and $\mathcal{V}$ are semiclassical linear functionals and they are again related by a rational factor. In addition, we will give differential equations satisfied by the formal Stieltjes series associated with $\mathcal{U}$ and $\mathcal{V}$, from which we can compute them.

On the other hand, when the above linear functionals are associated with positive Borel measures supported on the real line, then a useful algebraic relation between the sequences $\{S_n(x;\lambda)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ will be deduced, assuming that the measures form an (M, N)-coherent pair of order m and $\{S_n(x;\lambda)\}_{n\geq 0}$ is the SMOP with respect to the Sobolev inner product

$$\langle p(x), r(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)r(x)d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x)r^{(m)}(x)d\mu_1, \quad \lambda > 0,\ m \geq 1.$$

This is the topic to be analyzed in Section 2.2, and the cases of $(1,0)$ and $(1,1)$ -coherence of order m will be carefully studied. Furthermore, we will built and implement an efficient algorithm for the computation of the Fourier-Sobolev coefficients, i.e., the coefficients of the Fourier expansion of functions of the Sobolev space $W^{m,2}(I,\mu_0,\mu_1)$ in terms of the SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$. It is important to note that, from such an algorithm, the evaluation of the Fourier-Sobolev coefficients does not need the explicit expressions of the Sobolev orthogonal polynomials. At the end of this section an illustrative example of a Fourier-Sobolev expansion will be presented for a particular situation involving a $(2,1)$-coherent pair of order 3.

Finally, in Section 2.3, taking into account the monic Jacobi matrices associated with the regular linear functionals $\mathcal{U}$ and $\mathcal{V}$, we will study the (M,N)-coherence relation from a matrix point of view. In addition, the case when $\mathcal{U}$ is a classical linear functional and $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order m will also be analyzed. Besides, a matrix interpretation of the relation between (M,N)-coherent pairs of order m and Sobolev orthogonal polynomials will be studied.

2.1 (M,N)-Coherent Pairs of Order (m,k)

Let us recall that a pair of regular linear functionals $(\mathcal{U},\mathcal{V})$ is said to be a (M,N)-*coherent pair of order* (m,k), with M,N,m,k fixed nonnegative integer numbers, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ fulfill the following linear algebraic structure relation

$$P_n^{[m]}(x)+\sum_{i=1}^{M}a_{i,n}P_{n-i}^{[m]}(x)=Q_n^{[k]}(x)+\sum_{i=1}^{N}b_{i,n}Q_{n-i}^{[k]}(x),\quad n\geq 0, \tag{2.1.1}$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, and $a_{i,n}=b_{i,n}=0$ if $i>n$. Furthermore, when $k=0$, $(\mathcal{U},\mathcal{V})$ is said to be a (M,N)-*coherent pair of order* m.

Remark 2.1.1. When $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order (m,k) and either $\mathcal{U}$ or $\mathcal{V}$ is a classical linear functional, then $(\mathcal{U},\mathcal{V})$ can be regarded as a (M,N)-coherent pair of order $(0,k)$ or $(m,0)$, respectively, and thus they can be seen as a (N,M)-coherent pair of order k or a (M,N)-coherent pair of order m, respectively.

The next theorem generalizes results obtained in [4, 5, 25, 32, 33, 66, 73, 88, 90, 99] and improves several results stated in [52, 54, 79, 105] by giving a complete description of the semiclassical case in the framework of (M,N)-coherence of order (m,k).

Theorem 2.1.2. *Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-coherent pair of order (m,k) given by (2.1.1), with $m\geq k$, and $\det(\mathcal{L}_{M+N})\neq 0$, where $\mathcal{L}_{M+N}=[l_{i,j}]_{i,j=0}^{M+N-1}$ is the matrix of order*

$M+N$ with entries

$$l_{i,j}=\begin{cases} a_{j-i,j} & \text{if} \quad 0\le i\le N-1 \quad \text{and} \quad i\le j\le M+i, \\ b_{j-i+N,j} & \text{if} \quad N\le i\le M+N-1 \quad \text{and} \quad i-N\le j\le i, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1.2}$$

and the convention $a_{0,j_1}=b_{0,j_2}=1$ for $0\le j_1\le N-1$ and $0\le j_2\le M-1$. Then, there exist polynomials $\phi_{M+k+n}(x)$ and $\psi_{N+m+n}(x)$ of degrees $M+k+n$ and $N+m+n$, respectively, such that

$$D^{m-k}[\phi_{M+k+n}(x)\mathcal{V}]=\psi_{N+m+n}(x)\mathcal{U}, \quad n\ge 0, \tag{2.1.3}$$

and each one of the linear functionals $\mathcal{U}$ and $\mathcal{V}$ is a rational modification of the other one, i.e., there exist polynomials $\varphi(x)$ and $\rho(x)$ such that

$$\varphi(x)\mathcal{U}=\rho(x)\mathcal{V}. \tag{2.1.4}$$

Moreover,

(i) *If $m=k$, then $\mathcal{U}$ is a semiclassical linear functional if and only if so is $\mathcal{V}$.*

(ii) *If $m>k$, then $\mathcal{U}$ and $\mathcal{V}$ are semiclassical linear functionals.*

Proof. According to (2.1.1), let us consider the polynomial

$$R_n(x)=\sum_{i=0}^{M} a_{i,n}P_{n-i}^{[m]}(x)=\sum_{i=0}^{N} b_{i,n}Q_{n-i}^{[k]}(x), \quad n\ge 0, \tag{2.1.5}$$

where $a_{0,n}=b_{0,n}=1$. Let $\{\mathfrak{p}_n\}_{n\ge0}$, $\{\mathfrak{q}_n\}_{n\ge0}$, $\{\mathfrak{r}_n\}_{n\ge0}$, $\{\mathfrak{e}_{n,m}\}_{n\ge0}$, and $\{\mathfrak{h}_{n,k}\}_{n\ge0}$ be the dual bases associated with the SMOP $\{P_n(x)\}_{n\ge0}$, $\{Q_n(x)\}_{n\ge0}$ and the sequences $\{R_n(x)\}_{n\ge0}$, $\{P_n^{[m]}(x)\}_{n\ge0}$, and $\{Q_n^{[k]}(x)\}_{n\ge0}$, respectively. Since

$$\langle \mathfrak{e}_{n,m}, R_j(x)\rangle \overset{(2.1.5)}{=} \sum_{i=0}^{M}\langle \mathfrak{e}_{n,m}, a_{i,j}P_{j-i}^{[m]}(x)\rangle = \begin{cases} a_{j-n,j} & \text{if} \quad n\le j\le n+M, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle \mathfrak{h}_{n,k}, R_j(x)\rangle \overset{(2.1.5)}{=} \sum_{i=0}^{N}\langle \mathfrak{h}_{n,k}, b_{i,j}Q_{j-i}^{[k]}(x)\rangle = \begin{cases} b_{j-n,j} & \text{if} \quad n\le j\le n+N, \\ 0 & \text{otherwise}, \end{cases}$$

we get

$$\mathfrak{e}_{n,m}=\sum_{j\ge0}\langle \mathfrak{e}_{n,m}, R_j(x)\rangle \mathfrak{r}_j=\sum_{j=n}^{n+M} a_{j-n,j}\mathfrak{r}_j, \quad n\ge 0, \tag{2.1.6}$$

$$\mathfrak{h}_{n,k} = \sum_{j\geq 0} \langle \mathfrak{h}_{n,k}, R_j(x) \rangle \mathfrak{r}_j = \sum_{j=n}^{n+N} b_{j-n,j} \mathfrak{r}_j, \quad n \geq 0. \tag{2.1.7}$$

For $0 \leq n \leq N-1$ and $0 \leq n \leq M-1$, respectively, these equations yield the following system of linear equations

$$\mathcal{L}_{M+N} \begin{bmatrix} \mathfrak{r}_0 \\ \vdots \\ \mathfrak{r}_{N-1} \\ \mathfrak{r}_N \\ \vdots \\ \mathfrak{r}_{N+M-1} \end{bmatrix} = \begin{bmatrix} \mathfrak{e}_{0,m} \\ \vdots \\ \mathfrak{e}_{N-1,m} \\ \mathfrak{h}_{0,k} \\ \vdots \\ \mathfrak{h}_{M-1,k} \end{bmatrix},$$

where the matrix $\mathcal{L}_{M+N} = [l_{i,j}]_{i,j=0}^{M+N-1}$ is given by (2.1.2). Since by assumption, $\det(\mathcal{L}_{M+N}) \neq 0$, we can solve this linear system and get

$$\mathfrak{r}_i = \alpha_{i,0}\mathfrak{e}_{0,m} + \cdots + \alpha_{i,N-1}\mathfrak{e}_{N-1,m} + \alpha_{i,N}\mathfrak{h}_{0,k} + \cdots + \alpha_{i,N+M-1}\mathfrak{h}_{M-1,k}, \quad 0 \leq i \leq M+N-1, \tag{2.1.8}$$

where $\alpha_{i,j}$, $0 \leq j \leq N+M-1$, are some constants. On the other hand, for every $i \geq 0$, if we multiply (2.1.6) for $n = N+i$ by $b_{N,M+N+i}$, and (2.1.7) for $n = M+i$ by $a_{M,M+N+i}$, and then we subtract the resulting equations, we obtain

$$\begin{aligned} &b_{N,M+N+i}\mathfrak{e}_{N+i,m} - a_{M,M+N+i}\mathfrak{h}_{M+i,k} \\ &\qquad = \beta_{1,i}\mathfrak{r}_{\min\{M,N\}+i} + \cdots + \beta_{\max\{M,N\},i}\mathfrak{r}_{M+N+i-1}, \quad i \geq 0, \end{aligned} \tag{2.1.9}$$

where $\beta_{j,i}$, $1 \leq j \leq \max\{M,N\}$, $i \geq 0$, are constants. On the other hand, for $t \geq 0$ fixed, from (2.1.6) we can recursively get an expression for $\mathfrak{r}_{M+N+t}$ as a linear combination of $\mathfrak{r}_i$, $0 \leq i \leq M+N-1$, and $\mathfrak{e}_{j,m}$, $N \leq j \leq N+t$, (since $a_{M,M+j} \neq 0$, $N \leq j \leq N+t$). As a consequence and using (2.1.8), (2.1.9) becomes

$$\begin{aligned} &\widetilde{\alpha}_{i,0}\mathfrak{e}_{0,m} + \cdots + \widetilde{\alpha}_{i,N+i-1}\mathfrak{e}_{N+i-1,m} + b_{N,M+N+i}\mathfrak{e}_{N+i,m} \\ &\qquad = \widetilde{\beta}_{i,0}\mathfrak{h}_{0,k} + \cdots + \widetilde{\beta}_{i,M-1}\mathfrak{h}_{M-1,k} + a_{M,M+N+i}\mathfrak{h}_{M+i,k}, \quad i \geq 0, \end{aligned}$$

where $\widetilde{\alpha}_{i,j_1}$ and $\widetilde{\beta}_{i,j_2}$, for $0 \leq j_1 \leq N+i-1$ and $0 \leq j_2 \leq M-1$, are constants. Taking the mth derivative in the above equation, since $m \geq k$, from (1.3.4) it follows that

$$\begin{aligned} &\widehat{\alpha}_{i,0}\mathfrak{p}_m + \cdots + \widehat{\alpha}_{i,N+i-1}\mathfrak{p}_{N+i-1+m} + b_{N,M+N+i}(-1)^m(N+i+1)_m\mathfrak{p}_{N+i+m} = \\ &\qquad D^{m-k}\left[\widehat{\beta}_{i,0}\mathfrak{q}_k + \cdots + \widehat{\beta}_{i,M-1}\mathfrak{q}_{M-1+k} + a_{M,M+N+i}(-1)^k(M+i+1)_k\mathfrak{q}_{M+i+k}\right], \end{aligned}$$

for $i \geq 0$. Therefore, from (1.3.3) we get (2.1.3) with

$$\phi_{M+k+n}(x) = (-1)^k \frac{(M+n+1)_k a_{M,M+N+n}}{\langle \mathcal{V}, Q^2_{M+k+n}(x)\rangle} x^{M+k+n} + \textit{lower degree terms}, \ \ n \geq 0,$$
$$\psi_{N+m+n}(x) = (-1)^m \frac{(N+n+1)_m b_{N,M+N+n}}{\langle \mathcal{U}, P^2_{N+m+n}(x)\rangle} x^{N+m+n} + \textit{lower degree terms}, \ \ n \geq 0.$$

Notice that when $m = k$, for every $n \geq 0$, (2.1.3) becomes (2.1.4) with

$$\rho(x) = \phi_{M+k+n}(x) \quad \text{and} \quad \varphi(x) = \psi_{N+m+n}(x),$$

and, as a consequence, the statement *(i)* follows from Proposition 1.4.4.

On the other hand, from (1.1.6), (2.1.3) becomes

$$\sum_{i=0}^{m-k} \binom{m-k}{i} \phi^{(i)}_{M+k+n}(x) D^{m-k-i}\mathcal{V} = \psi_{N+m+n}(x)\mathcal{U}, \quad n \geq 0, \tag{2.1.10}$$

with $\deg(\phi_{M+k+n}(x)) = M+k+n$ and $\deg(\psi_{N+m+n}(x)) = N+m+n$. Hence, let us consider the following linear system resulting from (2.1.10) for $n = 0, 1, \ldots m-k$,

$$\mathcal{T}_{m-k+1}(x) \begin{bmatrix} D^{m-k}\mathcal{V} \\ \vdots \\ D\mathcal{V} \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} \psi_{N+m}(x)\mathcal{U} \\ \psi_{N+m+1}(x)\mathcal{U} \\ \vdots \\ \psi_{N+m+(m-k)}(x)\mathcal{U} \end{bmatrix},$$

where

$$\begin{aligned} \rho(x) &= \det\left(\mathcal{T}_{m-k+1}(x)\right) \\ &= \prod_{i=0}^{m-k} \binom{m-k}{i} W[\phi_{M+k}(x), \phi_{M+k+1}(x), \ldots, \phi_{M+k+(m-k)}(x)] \neq 0, \end{aligned}$$

and $W[\,\cdot\,]$ denotes the Wronskian. If $m > k$, we can solve this system for $\mathcal{V}$ and $D\mathcal{V}$ and thus (2.1.4) follows as well as

$$\rho(x)D\mathcal{V} = \varsigma(x)\mathcal{U},$$

where $\varphi(x)$ and $\varsigma(x)$ are some polynomials. As a consequence,

$$D\left[\varphi(x)\rho(x)\mathcal{V}\right] = \left(\varphi(x)\rho(x)\right)'\mathcal{V} + \varphi(x)\varsigma(x)\mathcal{U} = \left[\left(\varphi(x)\rho(x)\right)' + \varsigma(x)\rho(x)\right]\mathcal{V},$$
$$D\left[\varphi(x)\rho(x)\mathcal{U}\right] = D\left[\rho^2(x)\mathcal{V}\right] = 2\rho(x)\rho'(x)\mathcal{V} + \rho(x)\varsigma(x)\mathcal{U} = \left[2\varphi(x)\rho'(x) + \varsigma(x)\rho(x)\right]\mathcal{U}.$$

Therefore, $\mathcal{V}$ and $\mathcal{U}$ are semiclassical linear functionals, which proves the statement *(ii)*. □

Remark 2.1.3. When $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order (m,k) and $m=k$, we can not conclude that $\mathcal{U}$ and $\mathcal{V}$ are semiclassical. Indeed, in [105, Section 4] it was proved that if $\mathcal{U}$ and $\mathcal{V}$ are related by

$$\varphi(x)\mathcal{U}=\rho(x)\mathcal{V}, \quad \text{with} \quad \deg(\varphi(x))=N,\ \deg(\rho(x))=M,$$

then

$$\sum_{i=n-M}^{n+N} a_{i,n,1}P_i(x)=\sum_{i=n-N}^{n+N} b_{i,n,1}Q_i(x), \quad \text{and}$$
$$\sum_{i=n-M}^{n+M} a_{i,n,2}P_i(x)=\sum_{i=n-N}^{n+M} b_{i,n,2}Q_i(x), \quad \text{for } n\geq 0,$$

hold, where $\{a_{i,n,j}\}_{n\geq 0}$ and $\{b_{i,n,j}\}_{n\geq 0}$, $j=1,2$, are some constants. Thus, in this case, for any pair of nonzero polynomials $\varphi(x)$ and $\rho(x)$, we can choose either $\mathcal{U}$ or $\mathcal{V}$ being a non-semiclassical linear functional, and as a consequence, so is the other one.

Remark 2.1.4. When $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order (m,k) of positive definite linear functionals satisfying the same conditions of Theorem 2.1.2, then there exist polynomials $\varphi(x)$ and $\rho(x)$ such that $\mathcal{U}$ and $\mathcal{V}$ are related by

$$\varphi(x)\mathcal{U}=\rho(x)\mathcal{V}.$$

Therefore, when the zeros of either $\varphi(x)$ or $\rho(x)$ satisfy certain conditions, Proposition 1.3.5 states the relation between the positive Borel measures μ_0 and μ_1 corresponding to $\mathcal{U}$ and $\mathcal{V}$, respectively. More precisely, it gives an expression for either μ_0 in terms of μ_1, or μ_1 in terms of μ_0.

In the following theorem we deduce some relations for the formal Stieltjes series associated with the linear functionals constituting a (M,N)-coherent pair of order (m,k). Thus, we generalize the results in [54, Section 4].

Theorem 2.1.5. *If $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order (m,k) given by (2.1.1) and assuming the same condition as in Theorem 2.1.2, then*

$$\psi_{N+m+n}(z)S_{\mathcal{U}}(z)-\left[\phi_{M+k+n}(z)S_{\mathcal{V}}(z)\right]^{(m-k)}=A_n(z), \quad n\geq 0, \tag{2.1.11}$$

where

$$A_n(z)=\left(\mathcal{V}\theta_0\phi_{M+k+n}\right)^{(m-k)}(z)-\left(\mathcal{U}\theta_0\psi_{N+m+n}\right)(z),$$

with $\deg(A_n(z))\leq n-1+\max\{M+2k-m,N+m\}$, and $\left(\mathcal{U}\theta_0 p\right)(x)$, for $p\in\mathbb{P}$, is given by (1.1.2).

Moreover, $S_{\mathcal{V}}(z)$ is the (formal) solution of the following non-homogeneous ordinary differential equations of order $m-k$

$$\sum_{i=0}^{m-k} B_{i,n}(z) S_{\mathcal{V}}^{(i)}(z) = C_n(z), \quad n \geq 0, \tag{2.1.12}$$

with polynomial coefficients

$$B_{i,n}(z) = \binom{m-k}{i} \left[\psi_{N+m+n+1}(z) \phi_{M+k+n}^{(m-k-i)}(z) - \psi_{N+m+n}(z) \phi_{M+k+n+1}^{(m-k-i)}(z) \right],$$
$$C_n(z) = \psi_{N+m+n}(z) A_{n+1}(z) - \psi_{N+m+n+1}(z) A_n(z),$$

where $\deg(B_{i,n}(z)) \leq M+N+2k+2n+i+1$ *and* $\deg(C_n(z)) \leq 2n+N+\max\{M+2k, N+2m\}$.

Proof. From Theorem 2.1.2, for $n \geq 0$ there exist polynomials

$$\phi_{M+k+n}(x) = \sum_{j=0}^{M+k+n} r_{j,n} x^j \quad \text{and} \quad \psi_{N+m+n}(x) = \sum_{j=0}^{N+m+n} t_{j,n} x^j$$

such that

$$\left\langle D^{m-k}[\phi_{M+k+n}(x)\mathcal{V}], x^i \right\rangle = \left\langle \psi_{N+m+n}(x)\mathcal{U}, x^i \right\rangle, \quad i \geq 0.$$

So

$$(-1)^{m-k}(i-m+k+1)_{m-k} \sum_{j=0}^{M+k+n} r_{j,n} v_{i-m+k+j} = \sum_{j=0}^{N+m+n} t_{j,n} u_{i+j}, \quad i,n \geq 0,$$

where $v_{i-m+k+j} = 0$ if $i-m+k+j < 0$. Thus, multiplying the above expression by $z^{-(i+1)}$ and adding for $i = 0, 1, \ldots$, the left hand side becomes, taking into account (1.1.7) and (1.1.9),

$$\begin{aligned}
\sum_{i \geq m-k} (i-m+k+1)_{m-k} \sum_{j=0}^{M+k+n} r_{j,n} v_{i-m+k+j} z^{-(i+1)} \\
= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{\ell \geq 0} (\ell+1)_{m-k} \frac{v_{\ell+j}}{z^{\ell+j+1}} \\
= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{l=0}^{m-k} \binom{m-k}{l} (-j)_{m-k-l} z^l \sum_{\ell \geq 0} (\ell+1+j)_l \frac{v_{\ell+j}}{z^{\ell+j+1+l}} \\
= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{l=0}^{m-k} \binom{m-k}{l} (-j)_{m-k-l} z^l \left[(-1)^{l+1} S_{\mathcal{V}}^{(l)}(z) - \sum_{i=0}^{j-1} (i+1)_l \frac{v_i}{z^{i+1+l}} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k+l} \sum_{l=0}^{m-k} \binom{m-k}{l} (-1)^{m-k-l} (j-m+k+l+1)_{m-k-l} (-1)^{l+1} S_{\mathcal{V}}^{(l)}(z) \\
&- \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{i=0}^{j-1} (i+1-j)_{m-k} \frac{v_i}{z^{i+1}} = (-1)^{m-k+1} \sum_{l=0}^{m-k} \binom{m-k}{l} \phi_{M+k+n}^{(m-k-l)}(z) S_{\mathcal{V}}^{(l)}(z) \\
&\qquad - \sum_{j=0}^{M+k+n-1} r_{j+1,n} \sum_{i=0}^{j} (-1)^{m-k} \left((j-i)-m+k+1\right)_{m-k} v_i z^{j-i-m+k} \\
&\qquad = (-1)^{m-k-1} \left[\phi_{M+k+n}(z) S_{\mathcal{V}}(z)\right]^{(m-k)} + (-1)^{m-k-1} \left(\mathcal{V}\theta_0 \phi_{M+k+n}\right)^{(m-k)}(z).
\end{aligned}
$$

The right hand side reads

$$
\begin{aligned}
\sum_{i\geq 0} \left[\sum_{j=0}^{N+m+n} t_{j,n} u_{i+j}\right] z^{-(i+1)} &= \sum_{j=0}^{N+m+n} t_{j,n} z^j \left[-S_{\mathcal{U}}(z) - \sum_{i=0}^{j-1} \frac{u_i}{z^{i+1}}\right] \\
&= -\psi_{N+m+n}(z) S_{\mathcal{U}}(z) - \left(\mathcal{U}\theta_0 \psi_{N+m+n}\right)(z).
\end{aligned}
$$

Therefore, (2.1.11) follows. On the other hand, from (2.1.11) for n and $n+1$, we can obtain

$$
\begin{aligned}
&\psi_{N+m+n+1}(z) \left[\phi_{M+k+n}(z) S_{\mathcal{V}}(z)\right]^{(m-k)} - \psi_{N+m+n}(z) \left[\phi_{M+k+n+1}(z) S_{\mathcal{V}}(z)\right]^{(m-k)} \\
&\qquad = \psi_{N+m+n}(z) A_{n+1}(z) - \psi_{N+m+n+1}(z) A_n(z), \quad n \geq 0.
\end{aligned}
$$

Thus, using the Leibniz rule, (2.1.12) holds. □

Remark 2.1.6. We can get $S_{\mathcal{V}}$ if we solve (formally) any differential equation in (2.1.12). Consequently, from (2.1.11) we can also obtain $S_{\mathcal{U}}$.

2.2 Sobolev Orthogonal Polynomials and (M,N)-Coherent Pairs of Order m

In this section, $\mathbb{P}$ will denote the linear space of polynomials with real coefficients. We will also assume that $\mathcal{U}$ and $\mathcal{V}$ are positive definite linear functionals and μ_0 and μ_1 are the corresponding positive Borel measures supported on an infinite subset of the real line. Besides, we consider the Sobolev inner product

$$
\langle p(x), r(x)\rangle_\lambda = \int_{\mathbb{R}} p(x) r(x) d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x) r^{(m)}(x) d\mu_1, \quad p, r \in \mathbb{P}, \lambda > 0,\ m \in \mathbb{Z}^+, \tag{2.2.1}
$$

and the associated SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$. The completion of $\mathbb{P}$ with respect to the norm

$$
\|\cdot\|_\lambda = \langle \cdot, \cdot \rangle_\lambda^{1/2}
$$

yields the appropriate Sobolev space of functions. Notice that (2.2.1) can be rewritten as

$$\langle p(x), r(x)\rangle_{\lambda} = \langle p(x), r(x)\rangle_{\mu_0} + \lambda\langle p^{(m)}(x), r^{(m)}(x)\rangle_{\mu_1},$$

where $\langle\cdot\,,\cdot\rangle_{\mu_i}$ is the inner product induced by $d\mu_i$, $i=0,1$.

Remark 2.2.1. If $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$, and $\{S_n(x;\lambda)\}_{n\geq 0}$ are the SMOP with respect to μ_0, μ_1, and $\langle\cdot\,,\cdot\rangle_{\lambda}$, respectively, then

$$Q_n(x) = P_n^{[m]}(x) + \sum_{j=0}^{n-1} \frac{(j+1)_m}{(n+1)_m} \frac{\langle T_{n+m}(x), P_{j+m}(x)\rangle_{\mu_0}}{\|P_{j+m}\|^2_{\mu_0}} P_j^{[m]}(x), \quad n\geq 0, \tag{2.2.2}$$

$$S_n(x;\lambda) + \sum_{i=m}^{n-1} \frac{\langle T_n(x), S_i(x;\lambda)\rangle_{\mu_0}}{\|S_i\|^2_{\lambda}} S_i(x;\lambda) = P_n(x) + \sum_{i=m}^{n-1} \frac{\langle T_n(x), P_i(x)\rangle_{\mu_0}}{\|P_i\|^2_{\mu_0}} P_i(x), \tag{2.2.3}$$

for $n\geq m$. Notice that $S_n(x;\lambda) = P_n(x)$ for $n\leq m$, as well as we denote

$$T_n(x) = \lim_{\lambda\longrightarrow\infty} S_n(x;\lambda)\,, \quad n\geq 0. \tag{2.2.4}$$

Proof. From (2.2.1), $\langle P_n(x), x^i\rangle_{\lambda} = 0$, for $i<n<m$, and then $S_n(x;\lambda) = P_n(x)$ for $n<m$. Besides, the coefficients of the Sobolev MOP's $S_n(x;\lambda)$ are rational functions of λ. More precisely, their numerator and denominator are polynomials in λ of the same degree. Indeed, from the uniqueness of the SMOP with respect to the Sobolev inner product $\langle\cdot,\cdot\rangle_{\lambda}$, every $S_n(x;\lambda)$ can be written as

$$S_n(x;\lambda) = \frac{1}{\det\left([w_{i,j}]_{i,j=0}^{n-1}\right)} \begin{vmatrix} w_{0,0} & \cdots & w_{0,n-1} & w_{0,n} \\ \vdots & \ddots & \vdots & \vdots \\ w_{n-1,0} & \cdots & w_{n-1,n-1} & w_{n-1,n} \\ 1 & \cdots & x^{n-1} & x^n \end{vmatrix}, \; n\geq 1, \quad S_0(x;\lambda)=1,$$

where $w_{i,j} = \langle x^i, x^j\rangle_{\lambda} = u_{i+j} + \lambda(i-m+1)_m(j-m+1)_m v_{(i-m)+(j-m)}$, for $i,j\geq 0$. Additionally, notice that $[w_{i,j}]_{i,j=0}^{n}$ is a symmetric matrix for $n\geq m$, and it is a Hankel matrix for $n<m$ (it is the Hankel matrix associated with $\mathcal{U}$). Thus, there exist the monic polynomials given by (2.2.4). From (2.2.4) and (2.2.1) it follows that, for $n\geq 0$,

$$\langle T_n(x), x^i\rangle_{\mu_0} = 0,\; i<\min\{n,m\}, \quad \langle T_n^{(m)}(x), x^j\rangle_{\mu_1} = 0,\; j<n-m. \tag{2.2.5}$$

Indeed,

$$\langle T_n(x), x^i\rangle_{\mu_0} = \lim_{\lambda\longrightarrow\infty}\left[\langle S_n(x;\lambda), x^i\rangle_{\lambda} - \lambda\langle S_n^{(m)}(x;\lambda), (x^i)^{(m)}\rangle_{\mu_1}\right] = 0, \quad i<\min\{n,m\},$$

$$\langle T_n^{(m)}(x), (x^i)^{(m)}\rangle_{\mu_1} = \lim_{\lambda\longrightarrow\infty}\frac{1}{\lambda}\left[0 - \langle S_n(x;\lambda), x^i\rangle_{\mu_0}\right] = 0, \quad i<n.$$

Hence, from (2.2.5) we get

$$T_n(x) = \sum_{i=m}^{n} \frac{\langle T_n(x), P_i(x)\rangle_{\mu_0}}{\|P_i\|^2_{\mu_0}} P_i(x) = \sum_{j=0}^{n-m} \frac{\langle T_n(x), P_{j+m}(x)\rangle_{\mu_0}}{\|P_{j+m}\|^2_{\mu_0}} P_{j+m}(x), \quad n \geq m,$$

$$\frac{T^{(m)}_{n+m}(x)}{(n+1)_m} = \sum_{i=0}^{n} \frac{\langle T^{(m)}_{n+m}(x)/(n+1)_m, Q_i(x)\rangle_{\mu_1}}{\|Q_i\|^2_{\mu_1}} Q_i(x) = Q_n(x), \quad n \geq 0, \tag{2.2.6}$$

and, therefore, (2.2.2) follows. On the other hand, from (2.2.1) and (2.2.5) we obtain

$$T_n(x) = \sum_{i=0}^{n} \frac{\langle T_n(x), S_i(x;\lambda)\rangle_{\lambda}}{\|S_i\|^2_{\lambda}} S_i(x;\lambda) = S_n(x;\lambda) + \sum_{i=m}^{n-1} \frac{\langle T_n(x), S_i(x;\lambda)\rangle_{\mu_0}}{\|S_i\|^2_{\lambda}} S_i(x;\lambda),$$

for $n \geq 0$, and, as a consequence, (2.2.3) holds. □

Recall that the pair of measures (μ_0, μ_1) is said to be a (M, N)-coherent pair of order m if it is a (M, N)-coherent pair of order $(m, 0)$, i.e, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$P^{[m]}_n(x) + \sum_{i=1}^{M} a_{i,n} P^{[m]}_{n-i}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0, \tag{2.2.7}$$

where $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$.

The following Theorem extends a fundamental algebraic property known for $(1,0)$-coherent, $(2,0)$-coherent, $(M+1,0)$-coherent, $(1,1)$-coherent and (M,N)-coherent pairs of measures, stated in [27, 33, 48, 55, 71], to (M,N)-coherent pairs of order m.

Theorem 2.2.2. *Let (μ_0, μ_1) be a (M, N)-coherent pair of order m given by (2.2.7), and $K = \max\{M, N\}$. Then, $S_n(x;\lambda) = P_n(x)$ for $n < m$ and*

$$P_{n+m}(x) + \sum_{i=1}^{M} \frac{(n+1)_m a_{i,n}}{(n-i+1)_m} P_{n-i+m}(x) = S_{n+m}(x;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda} S_{n-j+m}(x;\lambda), \quad n \geq 0, \tag{2.2.8}$$

where $c_{j,n,\lambda} = 0$ for $n < j \leq K$, and

$$c_{j,n,\lambda} = \frac{(n+1)_m}{\|S_{n-j+m}\|^2_{\lambda}} \left[\sum_{i=j}^{M} \frac{a_{i,n}}{(n-i+1)_m} \langle P_{n-i+m}(x), S_{n-j+m}(x;\lambda)\rangle_{\mu_0} \right.$$
$$\left. + \lambda \sum_{i=j}^{N} b_{i,n} \left\langle Q_{n-i}(x), S^{(m)}_{n-j+m}(x;\lambda) \right\rangle_{\mu_1} \right], \quad 1 \leq j \leq K. \tag{2.2.9}$$

Furthermore, for every $n \geq K$,

(i) *if* $M > N$ *and* $a_{M,n} \neq 0$, *then* $c_{K,n,\lambda} \neq 0$,

(ii) *if* $M < N$ *and* $b_{N,n} \neq 0$, *then* $c_{K,n,\lambda} \neq 0$,

(iii) *if* $M = N (= K)$ *and* $a_{M,n} b_{N,n} \neq 0$ *then,*

$$c_{K,n,\lambda} \neq 0 \quad \textit{iff} \quad a_{K,n}\|P_{n-K+m}\|^2_{\mu_0} + \lambda(n-K+1)^2_m b_{K,n}\|Q_{n-K}\|^2_{\mu_1} \neq 0.$$

Conversely, if (2.2.8) *holds for some constants* $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, *and* $\{a_{i,n}\}_{n\geq 0}$, $1 \leq i \leq M$, *such that* $c_{j,n,\lambda} = 0$, *when* $n - j + m < 0$, *and* $a_{i,n} = 0$, *when* $n - i + m < 0$, *then* (μ_0, μ_1) *is a* (M,K)*-coherent pair of order* m *given by*

$$P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x) = Q_n(x) + \sum_{j=1}^{K} b_{j,n} Q_{n-j}(x), \quad n \geq 0, \tag{2.2.10}$$

where $b_{j,n} = 0$ *for* $n < j \leq K$, *and*

$$b_{j,n} = \frac{\left\langle P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x), Q_{n-j}(x) \right\rangle_{\mu_1}}{\|Q_{n-j}\|^2_{\mu_1}}, \quad 1 \leq j \leq \min\{K, n\}, \; n \geq 0, \tag{2.2.11}$$

provided that $b_{K,n} \neq 0$, $n \geq K$, *hold.*

Proof. Since $\langle P_n(x), x^i \rangle_\lambda = 0$ for $i < n < m$, then $S_n(x;\lambda) = P_n(x)$ for $n < m$. On the other hand, substituting (2.2.6) in (2.2.7), and integrating m times both sides of the resulting equation, we get

$$\frac{P_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{M} a_{i,n} \frac{P_{n-i+m}(x)}{(n-i+1)_m} = \frac{T_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x)}{(n-i+1)_m} + \sum_{j=0}^{m-1} \kappa_{n,j} x^j, \quad n \geq 0.$$

Applying $\langle \cdot, x^i \rangle_{\mu_0}$, $i < m$, and taking into account (2.2.5), we obtain for every fixed $n \geq 0$, the system of linear equations

$$\sum_{j=0}^{m-1} \kappa_{n,j} u_{j+i} = 0, \quad i = 0, \ldots, m-1.$$

Since $\det\left([u_{i+j}]_{i,j=0}^{m-1}\right) \neq 0$, then $\kappa_{n,j} = 0$, for $j = 0, \ldots, m-1$ and $n \geq 0$. Therefore

$$\frac{P_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{M} a_{i,n} \frac{P_{n-i+m}(x)}{(n-i+1)_m} = \frac{T_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x)}{(n-i+1)_m}, \quad n \geq 0. \tag{2.2.12}$$

On the other hand,

$$\frac{T_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x)}{(n-i+1)_m} = \frac{S_{n+m}(x;\lambda)}{(n+1)_m} + \sum_{j=1}^{n+m} \frac{c_{j,n,\lambda}}{(n+1)_m} S_{n-j+m}(x;\lambda), \quad n \geq 0,$$

where from (2.2.1), (2.2.12) and (2.2.6), for $1 \leq j \leq n+m$, we get

$$\begin{aligned}\|S_{n-j+m}\|_\lambda^2 \frac{c_{j,n,\lambda}}{(n+1)_m} &= \sum_{i=1}^{M} \frac{a_{i,n}}{(n-i+1)_m} \langle P_{n-i+m}(x), S_{n-j+m}(x;\lambda)\rangle_{\mu_0} \\ &\quad + \lambda \sum_{i=1}^{N} b_{i,n} \left\langle Q_{n-i}(x), S_{n-j+m}^{(m)}(x;\lambda)\right\rangle_{\mu_1}.\end{aligned}$$

Then $c_{j,n,\lambda} = 0$ for $j > i$ or $j > \max\{M,N\} = K$. Therefore, (2.2.8) and (2.2.9) hold. Besides, for $n \geq K$,

$$\frac{c_{K,n,\lambda}}{(n+1)_m} = \frac{\frac{a_{M,n}}{(n-M+1)_m}\|P_{n-M+m}\|_{\mu_0}^2 \delta_{M,K} + \lambda(n-N+1)_m b_{N,n}\|Q_{n-N}\|_{\mu_1}^2 \delta_{N,K}}{\|S_{n-K+m}\|_\lambda^2},$$

from which *(i)*, *(ii)* and *(iii)* are deduced.

Finally, applying $\langle \cdot, p(x)\rangle_\lambda$ to both sides of (2.2.8), for any $p \in \mathbb{P}_{n-K+m-1}$, we get

$$0 = \lambda \left\langle P_{n+m}^{(m)}(x) + \sum_{i=1}^{M} \frac{(n+1)_m a_{i,n}}{(n-i+1)_m} P_{n-i+m}^{(m)}(x), p^{(m)}(x) \right\rangle_{\mu_1}, \quad p \in \mathbb{P}_{n-K+m-1},$$

and since each polynomial of degree $n-K-1$ is the m-th derivative of some polynomial of degree $n-K+m-1$, then

$$0 = \left\langle P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x), r(x) \right\rangle_{\mu_1}, \quad r \in \mathbb{P}_{n-K-1}.$$

Besides,

$$P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x) = Q_n(x) + \sum_{j=1}^{n} b_{j,n} Q_{n-j}(x), \quad n \geq 0,$$

where $b_{j,n}$ for $1 \leq j \leq n$, is given by (2.2.11). Therefore, (2.2.10) follows. □

Remark 2.2.3. Using Theorem 2.2.2, we can recursively compute the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$ and the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, when (μ_0, μ_1) is a (M,N)-coherent pair of order m and the coherence relation (2.2.7) is known. In the next section, we will deduce an algorithm (Algorithm 2.2.8) which allows to compute the Sobolev norms $\{\|S_n\|_\lambda^2\}_{n\geq 0}$ and the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, and, as a consequence, from (2.2.8) and $S_n(x;\lambda) = P_n(x)$ for $n < m$, we can get the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$.

2.2.1 Computation of the Fourier-Sobolev Coefficients for (M,N)-Coherent Pairs of Order m of Measures

Let I be a open interval of the real line and let $W^{m,2}[I,\mu_0,\mu_1]$ be the Sobolev space of smooth functions

$$W^{m,2}[I,\mu_0,\mu_1] = \left\{ f : I \to \mathbb{R} \,|\, f \in L^2_{\mu_0}(I)\,,\; f^{(m)} \in L^2_{\mu_1}(I) \right\}.$$

Every function $f \in W^{m,2}[I,\mu_0,\mu_1]$ has the following Fourier-Sobolev expansion with respect to the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} S_n(x;\lambda), \tag{2.2.13}$$

where

$$f_n \equiv f_n(\lambda) = \langle f(x), S_n(x;\lambda)\rangle_\lambda, \quad \text{and} \quad s_n \equiv s_n(\lambda) = \|S_n\|^2_\lambda, \quad n \geq 0. \tag{2.2.14}$$

In [48], [66] and [55] an efficient algorithm for computing the Fourier-Sobolev coefficients f_n/s_n, $n \geq 0$, when (μ_0,μ_1) is a $(1,0)$-coherent, $(2,0)$-coherent, and (M,N)-coherent pair, respectively, is done. Here we extend these algorithms to the general case when (μ_0,μ_1) is a (M,N)-coherent pair of order m. For this purpose, first we show how to compute the sequences $\{f_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ based on the algebraic property stated in Theorem 2.2.2. Finally, the algorithm will be a direct consequence of these results.

We use the following notation

$$\widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n} \quad \text{and} \quad \widetilde{b}_{i,n} = (n+1)_m b_{i,n}, \quad n \geq 0, \tag{2.2.15}$$

where $\widetilde{a}_{i,n} = \widetilde{b}_{i,n} = 0$ when $i > n$, and $\widetilde{a}_{0,n} = 1$ and $\widetilde{b}_{0,n} = (n+1)_m$ for $n \geq 0$, (since $a_{i,n} = b_{i,n} = 0$ for $i > n$, and $a_{0,n} = b_{0,n} = 1$ for $n \geq 0$).

Theorem 2.2.4. *Let (μ_0,μ_1) be a (M,N)-coherent pair of order m given by (2.2.7) and $K = \max\{M,N\}$. Then the sequence $\{f_n\}_{n\geq 0}$, given by (2.2.14), satisfies the following non-homogeneous linear difference equation of order K*

$$f_{n+m} + \sum_{j=1}^{K} c_{j,n,\lambda} f_{n-j+m} = \varrho_n, \quad n \geq 0, \tag{2.2.16}$$

where $c_{j,n,\lambda}$ is given by (2.2.9) and ϱ_n is defined by

$$\varrho_n \equiv \varrho_n(\lambda; f) = \left\langle f(x), \sum_{i=0}^{M} \widetilde{a}_{i,n} P_{n-i+m}(x) \right\rangle_{\mu_0} + \lambda \left\langle f^{(m)}(x), \sum_{i=0}^{N} \widetilde{b}_{i,n} Q_{n-i}(x) \right\rangle_{\mu_1}. \tag{2.2.17}$$

Proof. Applying $\langle f(x), \cdot \rangle_\lambda$ to both sides of (2.2.8) and using (2.2.1), (2.2.7), and (2.2.15), we get the desired result. $\square$

Now, we will show that the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, together with the Sobolev norms $\{s_n\}_{n\geq 0}$ satisfy the system of $K+1$ difference equations given by (2.2.18) and (2.2.21), with initial conditions $s_n = \|P_n\|^2_{\mu_0}$, $0 \leq n < m$, and $c_{j,n,\lambda} = 0$, $0 \leq n < j \leq K$, from which they can be computed. Besides, since the sequence $\{\varrho_n\}_{n\geq 0}$ can be directly computed in terms of the data (the (M,N)-coherence relation (2.2.7), the parameter λ, and the function f), then using (2.2.16), we can recursively compute the sequence $\{f_n\}_{n\geq 0}$. Thus, the sequences $\{f_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ will be deduced and, therefore, we get the Fourier-Sobolev coefficients $\{f_n/s_n\}_{n\geq 0}$.

Theorem 2.2.5. *The following relations hold*

$$s_{n-K+\ell+m}c_{K-\ell,n,\lambda} + \sum_{i=1}^{\ell} c_{i,n-K+\ell,\lambda}c_{K-\ell+i,n,\lambda}s_{n-K+\ell-i+m} = \zeta_{\ell,n,\lambda}, \quad 0 \leq \ell \leq K, n \geq 0, \tag{2.2.18}$$

where $c_{j,n,\lambda} = 0$ *for* $n < j \leq K$, *and*

$$\zeta_{\ell,n,\lambda} = \sum_{i=K-\ell}^{M} \widetilde{a}_{i,n}\widetilde{a}_{i-K+\ell,n-K+\ell}\|P_{n-i+m}\|^2_{\mu_0} + \lambda \sum_{i=K-\ell}^{N} \widetilde{b}_{i,n}\widetilde{b}_{i-K+\ell,n-K+\ell}\|Q_{n-i}\|^2_{\mu_1}. \tag{2.2.19}$$

Proof. From (2.2.8), (2.2.7), and taking into account (2.2.15), for $j = K-\ell$ and $0 \leq \ell \leq K$, $n \geq 0$, (2.2.9) becomes

$$\begin{aligned} s_{n-K+\ell+m}c_{K-\ell,n,\lambda} = & \sum_{i=K-\ell}^{M}\sum_{j=0}^{M} \widetilde{a}_{i,n}\widetilde{a}_{j,n-K+\ell} \left\langle P_{n-i+m}(x), P_{n-K+\ell-j+m}(x)\right\rangle_{\mu_0} \\ & - \sum_{i=K-\ell}^{M}\sum_{j=1}^{K} \widetilde{a}_{i,n}c_{j,n-K+\ell,\lambda} \left\langle P_{n-i+m}(x), S_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_0} \\ & + \lambda \sum_{i=K-\ell}^{N}\sum_{j=0}^{N} \widetilde{b}_{i,n}\widetilde{b}_{j,n-K+\ell} \left\langle Q_{n-i}(x), Q_{n-K+\ell-j}(x)\right\rangle_{\mu_1} \\ & - \lambda \sum_{i=K-\ell}^{N}\sum_{j=1}^{K} \widetilde{b}_{i,n}c_{j,n-K+\ell,\lambda} \left\langle Q_{n-i}(x), S^{(m)}_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_1}. \end{aligned} \tag{2.2.20}$$

Notice that, from orthogonality, the first and the third terms in the right-hand side of (2.2.20) are equal to

$$\sum_{i=K-\ell}^{M} \widetilde{a}_{i,n}\widetilde{a}_{i-K+\ell,n-K+\ell}\|P_{n-i+m}\|^2_{\mu_0} \quad \text{and} \quad \lambda \sum_{i=K-\ell}^{N} \widetilde{b}_{i,n}\widetilde{b}_{i-K+\ell,n-K+\ell}\|Q_{n-i}\|^2_{\mu_1},$$

respectively. On the other hand, the second and the fourth terms are equal to

$$-\sum_{j=1}^{\ell} c_{j,n-K+\ell,\lambda} \sum_{i=K-\ell+j}^{M} \widetilde{a}_{i,n} \langle P_{n-i+m}(x), S_{n-K+\ell-j+m}(x;\lambda)\rangle_{\mu_0} \quad \text{and}$$

$$-\lambda \sum_{j=1}^{\ell} c_{j,n-K+\ell,\lambda} \sum_{i=K-\ell+j}^{N} \widetilde{b}_{i,n} \langle Q_{n-i}(x), S^{(m)}_{n-K+\ell-j+m}(x;\lambda)\rangle_{\mu_1},$$

respectively. Indeed, since $\langle P_{n-i+m}(x), S_{n-K+\ell-j+m}(x;\lambda)\rangle_{\mu_0} = 0$ if $i < K-\ell+j$ or if $K-\ell+j > M$ (because $i \leq M$), then the second term in (2.2.20) is equal to

$$\sum_{j=1}^{M-K+\ell} \sum_{i=K-\ell+j}^{M} \widetilde{a}_{i,n} c_{j,n-K+\ell,\lambda} \left\langle P_{n-i+m}(x), S_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_0}$$
$$= \sum_{j=1}^{\ell} \sum_{i=K-\ell+j}^{M} \widetilde{a}_{i,n} c_{j,n-K+\ell,\lambda} \left\langle P_{n-i+m}(x), S_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_0},$$

where the last equality follows from $\sum_{j=M-K+\ell+1}^{\ell} \sum_{i=K-\ell+j}^{M} (\,\cdot\,) = 0$. In the same way we can obtain the fourth term. Furthermore, notice that from (2.2.9) the sum of the second and the fourth terms is $-\sum_{j=1}^{\ell} c_{j,n-K+\ell,\lambda} s_{n-K+\ell-j+m} c_{K-\ell+j,n,\lambda}$. Therefore, 2.2.18 follows. □

Corollary 2.2.6. *The sequence $\{s_n\}_{n\geq 0}$ satisfies the non-homogeneous linear difference equation of order K*

$$s_{n+m} + \sum_{i=1}^{K} c^2_{i,n,\lambda} s_{n-i+m} = \zeta_{K,n,\lambda}, \quad n \geq 0, \tag{2.2.21}$$

where $c_{j,n,\lambda} = 0$ for $n < j \leq K$, and

$$\zeta_{K,n,\lambda} = \sum_{i=0}^{M} \widetilde{a}^2_{i,n} \|P_{n-i+m}\|^2_{\mu_0} + \lambda \sum_{i=0}^{N} \widetilde{b}^2_{i,n} \|Q_{n-i}\|^2_{\mu_1}.$$

Proof. The proof is a straightforward consequence of (2.2.18) for $\ell = K$ (since (2.2.8) and (2.2.9) hold taking $c_{0,n,\lambda} = 1$ for $n \geq 0$). □

Additionally, from the (M,N)-coherence of order m, we can find bounds for $\{s_n\}_{n\geq 0}$, the norms of the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$, as follows

Corollary 2.2.7. *For $n \geq m$, the following inequalities hold*

$$\|P_n\|^2_{\mu_0} + \lambda(n+1-m)^2_m \|Q_{n-m}\|^2_{\mu_1} \leq s_n \leq \sum_{i=0}^{M} \widetilde{a}^2_{i,n-m} \|P_{n-i}\|^2_{\mu_0} + \lambda \sum_{i=0}^{N} \widetilde{b}^2_{i,n-m} \|Q_{n-m-i}\|^2_{\mu_1}. \tag{2.2.22}$$

Proof. From the extremal properties for monic Sobolev orthogonal and standard monic orthogonal polynomials, respectively, we get

$$s_n = \|S_n\|^2_{\mu_0} + \lambda\|S_n^{(m)}\|^2_{\mu_1} \geq \|P_n\|^2_{\mu_0} + \lambda(n+1-m)^2_m\|Q_{n-m}\|^2_{\mu_1}, \quad n \geq 0.$$

On the other hand, from (2.2.21) and since $\zeta_{K,n,\lambda} > 0$ for every $n \geq 0$, it follows that $s_{n+m} \leq \zeta_{K,n,\lambda}$, for $n \geq 0$. Substituting n by $n-m$, the proof is complete. □

Finally, replacing in (2.2.18) ℓ by $K-j$ and n by $n+j$, we get

$$s_{n+m}c_{j,n+j,\lambda} = \zeta_{K-j,n+j,\lambda} - \sum_{i=1}^{K-j} c_{i,n,\lambda}c_{j+i,n+j,\lambda}s_{n+m-i}, \quad 0 \leq j \leq K, n \geq 0. \tag{2.2.23}$$

These previous equations, $\|S_n\|_\lambda = \|P_n\|_{\mu_0}$ for $n < m$, and $c_{j,n,\lambda} = 0$ for $n < j \leq K$, allow us to compute all the Sobolev norms $\{s_n\}_{n\geq 0}$ as well as all the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, in the algebraic relation (2.2.8), as it is shown in the following algorithm. Additionally, using (2.2.16) and $f_n = \langle f(x), P_n(x)\rangle_{\mu_0}$ for $n < m$, we can compute the coefficients $\{f_n\}_{n\geq 0}$. Finally, as a consequence, it is possible to compute the Fourier-Sobolev coefficients $\{f_n/s_n\}_{n\geq 0}$ appearing in (2.2.13) for any function $f \in W^{m,2}[I, \mu_0, \mu_1]$.

Algorithm 2.2.8. This algorithm allows us to compute the Fourier-Sobolev coefficients $\{\hbar_{n,\lambda} = f_n/s_n\}_{n\geq 0}$ in (2.2.13) for a given function $f \in W^{m,2}[I, \mu_0, \mu_1]$, as well as the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, in (2.2.8), when (μ_0, μ_1) is a (M, N)-coherent pair of order m given by (2.2.7).

- *Starting Data:* The initial conditions are $K = \max\{M, N\}$ and

  $$c_{j,n,\lambda} = 0, \; j > K \text{ or } n < j \leq K, n \geq 0; \quad c_{0,n,\lambda} = 1, \; n \geq 0;$$
  $$s_n = \|P_n\|^2_{\mu_0}, \quad f_n = \langle f(x), P_n(x)\rangle_{\mu_0}, \quad \hbar_{n,\lambda} = f_n/s_n, \quad 0 \leq n < m.$$

  Furthermore, we must take into account the expression for ϱ_n and $\zeta_{j,n,\lambda}$, $0 \leq j \leq K$, and $n \geq 0$, given by (2.2.17) and (2.2.19), respectively. (See also (2.2.15)).

- *Step n, for every $n \geq 0$ fixed:* From the Starting Data and the information obtained in the Steps 1 to $n-1$, we can compute

  (i) s_{m+n} from (2.2.23) taking $j = 0$, and the elements $c_{j,n+j,\lambda}$ for $j = 1, \cdots, K$,

  (ii) f_{m+n} from (2.2.16),

  (iii) and the Fourier-Sobolev coefficient $\hbar_{n,\lambda}$

  as follows

  $$s_{m+n} = \zeta_{K,n,\lambda} - \sum_{i=1}^{\min\{K,n\}} c^2_{i,n,\lambda}s_{m+n-i};$$

$$c_{j,n+j,\lambda} = \left(\zeta_{K-j,n+j,\lambda} - \sum_{i=1}^{\min\{K-j,n\}} c_{i,n,\lambda} c_{i+j,n+j,\lambda} s_{m+n-i}\right) / s_{m+n}, \quad 1 \leq j \leq K;$$
$$f_{m+n} = \varrho_n - \sum_{i=1}^{\min\{K,n\}} c_{i,n,\lambda} f_{m+n-i};$$
$$\hbar_{m+n,\lambda} = f_{m+n}/s_{m+n}.$$

Remark 2.2.9. Notice that the computation of the Sobolev norms $\{s_{m+n}\}_{n\geq 0}$ and the coefficients $\{c_{j,n+j,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, follows the scheme illustrated by the next matrix with $K+1$ rows and infinitely many columns, where the computation must be done successively along the decreasing diagonals

$$\begin{bmatrix} s_m & s_{m+1} & s_{m+2} & \cdots & & & & \\ 0 & c_{1,1,\lambda} & c_{1,2,\lambda} & c_{1,3,\lambda} & \cdots & & & \\ 0 & 0 & c_{2,2,\lambda} & c_{2,3,\lambda} & c_{2,4,\lambda} & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & c_{K,K,\lambda} & c_{K,K+1,\lambda} & c_{K,K+2,\lambda} & \cdots \end{bmatrix}$$

Remark 2.2.10. As a consequence of Algorithm 2.2.8, the computation of the Fourier-Sobolev coefficients does not need to know explicitly the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$, when (μ_0, μ_1) is a (M,N)-coherent pair of order m. However, to get the Fourier-Sobolev series, we can recursively compute the Sobolev SMOP using (2.2.8) and $S_n(x;\lambda) = P_n(x)$ for $n < m$, because the Sobolev norms $\{s_{m+n}\}_{n\geq 0}$ and the coefficients $\{c_{j,n+j,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, already were obtained from the Algorithm 2.2.8.

2.2.2 Two Particular Cases

Some sequences involved in Algorithm 2.2.8 satisfy additional properties when (μ_0, μ_1) is a $(1,1)$-coherent or $(1,0)$-coherent pair of order m.

Theorem 2.2.11. *Let (μ_0, μ_1) be a $(1,1)$-coherent pair of order m with corresponding SMOP $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$, given by*

$$P_n^{[m]}(x) + a_{1,n} P_{n-1}^{[m]}(x) = Q_n(x) + b_{1,n} Q_{n-1}(x), \quad n \geq 0,$$

where $a_{1,0} = b_{1,0} = 0$. Then,

(i) *The Sobolev SMOP with respect to the inner product (2.2.1), $\{S_n(x;\lambda)\}_{n\geq 0}$, satisfies $S_n(x;\lambda) = P_n(x)$ for $n < m$, and*

$$P_{m+n}(x) + \frac{m+n}{n} a_{1,n} P_{m+n-1}(x) = S_{m+n}(x;\lambda) + c_{1,n,\lambda} S_{m+n-1}(x;\lambda), \quad n \geq 0, \tag{2.2.24}$$

where $c_{1,0,\lambda} = 0$ *and*

$$c_{1,n,\lambda} s_{m+n-1} = \frac{m+n}{n} a_{1,n} \|P_{m+n-1}\|^2_{\mu_0} + \lambda (n)_m (n+1)_m b_{1,n} \|Q_{n-1}\|^2_{\mu_1}.$$

(ii) *The sequences of Sobolev norms* $\{s_n\}_{n\geq 0}$, *with* $s_n = \|S_n\|^2_\lambda$, *and constants* $\{c_{1,n,\lambda}\}_{n\geq 0}$ *in* (2.2.24) *can be computed by*

$$s_{m+n+1} = \zeta_{1,n+1,\lambda} - \frac{\zeta^2_{0,n+1,\lambda}}{s_{m+n}} \quad \text{and} \quad \frac{\zeta_{0,n+1,\lambda}}{c_{1,n+1,\lambda}} = \zeta_{1,n,\lambda} - \zeta_{0,n,\lambda} c_{1,n,\lambda}, \quad n \geq 0, \tag{2.2.25}$$

(the previous equation holds if $\zeta_{0,n+1,\lambda} \neq 0$ *for* $n \geq 0$*), and*

$$c_{1,n+1,\lambda} = \frac{\zeta_{0,n+1,\lambda}}{s_{n+m}}, \quad n \geq 0,$$

holds, with initial conditions $c_{1,0,\lambda} = 0$, $s_m = \zeta_{1,0,\lambda}$, *and* $s_n = \|P_n\|_{\mu_0}$ *for* $n < m$, *where*

$$\zeta_{0,n,\lambda} = \frac{n+m}{n} a_{1,n} \|P_{m+n-1}\|^2_{\mu_0} + \lambda (n)_m (n+1)_m b_{1,n} \|Q_{n-1}\|^2_{\mu_1}, \tag{2.2.26}$$

$$\zeta_{1,n,\lambda} = \|P_{n+m}\|^2_{\mu_0} + \frac{(n+m)^2}{n^2} a^2_{1,n} \|P_{n+m-1}\|^2_{\mu_0} + \lambda (n+1)^2_m \big[\|Q_n\|^2_{\mu_1} + b^2_{1,n} \|Q_{n-1}\|^2_{\mu_1}\big].$$

Furthermore,

$$\begin{aligned} \|P_n\|^2_{\mu_0} + \lambda (n+1)^2_m \|Q_{n-m}\|^2_{\mu_1} \leq s_n \leq \|P_n\|^2_{\mu_0} + \frac{n^2}{(n-m)^2} a^2_{1,n-m} \|P_{n-1}\|^2_{\mu_0} \\ + \lambda (n-m+1)^2_m \big[\|Q_{n-m}\|^2_{\mu_1} + b^2_{1,n-m} \|Q_{n-m-1}\|^2_{\mu_1}\big], \quad n \geq 0. \end{aligned} \tag{2.2.27}$$

(iii) *If* $\zeta_{0,n+1,\lambda} \neq 0$ *for* $n \geq 0$, *then every Sobolev norm* s_{m+n} *for* $n \geq 0$, *and each constant* $c_{1,n,\lambda}$ *for* $n \geq 1$, *can be represented, respectively, by the continued fraction*

$$s_{m+n} = \frac{\zeta^2_{0,n+1,\lambda}}{|\zeta_{1,n+1,\lambda}} - \frac{\zeta^2_{0,n+2,\lambda}|}{|\zeta_{1,n+2,\lambda}} - \cdots, \quad n \geq 0,$$

$$c_{1,n,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}} - \frac{\frac{\zeta_{0,n+1,\lambda}}{\zeta_{0,n,\lambda}}}{\left|\frac{\zeta_{1,n+1,\lambda}}{\zeta_{0,n+1,\lambda}}\right.} - \frac{\left.\frac{\zeta_{0,n+2,\lambda}}{\zeta_{0,n+1,\lambda}}\right|}{\left|\frac{\zeta_{1,n+2,\lambda}}{\zeta_{0,n+2,\lambda}}\right.} - \cdots, \quad n \geq 1.$$

(iv) *If* $\zeta_{0,n+1,\lambda} \neq 0$ *for* $n \geq 0$, *then the Sobolev norms* $\{s_n\}_{n\geq 0}$ *and the constants* $\{c_{1,n,\lambda}\}_{n\geq 0}$ *in* (2.2.24) *are*

$$s_{m+n} = \frac{\varpi_{n+1}(0;\lambda)}{\varpi_n(0;\lambda)}, \quad c_{1,n+1,\lambda} = \zeta_{0,n+1,\lambda} \frac{\varpi_n(0;\lambda)}{\varpi_{n+1}(0;\lambda)}, \quad n \geq 0, \tag{2.2.28}$$

where $\{\varpi_n(x;\lambda)\}_{n\geq 0}$ *is a SMOP with respect to some positive definite linear functional, satisfying the TTRR:* $\varpi_0(x;\lambda)=1$, $\varpi_{-1}(x;\lambda)=0$,

$$\varpi_{n+1}(x;\lambda) = (x+\zeta_{1,n,\lambda})\varpi_n(x;\lambda) - \zeta^2_{0,n,\lambda}\varpi_{n-1}(x;\lambda), \quad n\geq 0. \tag{2.2.29}$$

(v) *Let* $f\in W^{m,2}[I,\mu_0,\mu_1]$ *and let* $\sum_{n=0}^{\infty}\frac{f_n}{s_n}S_n(x;\lambda)$ *be its Fourier-Sobolev series. Then, the Fourier-Sobolev coefficients* $\{f_n/s_n\}_{n\geq 0}$ *can be computed using* (2.2.25) *and*

$$f_{m+n} = \varrho_n - c_{1,n,\lambda}f_{m+n-1}, \quad n\geq 0,$$

where

$$\varrho_n = \left\langle f(x), P_{n+m} + \frac{n+m}{n}a_{1,n}P_{n+m-1}\right\rangle_{\mu_0} + \lambda(n+1)_m\left\langle f^{(m)}(x), Q_n + b_{1,n}Q_{n-1}\right\rangle_{\mu_1}.$$

Proof. ***(i)*** It is immediate from Theorem 2.2.2.

(ii) (2.2.23) becomes

$$s_{n+m} = \zeta_{1,n,\lambda} - c^2_{1,n,\lambda}s_{n+m-1}, \quad \text{and} \quad s_{n+m}c_{1,n+1,\lambda} = \zeta_{0,n+1,\lambda}, \quad n\geq 0.$$

As a consequence, (2.2.25) holds. Besides, (2.2.26) follows from (2.2.15) and (2.2.19).

On the other hand, the lower and upper bounds for s_n, $n\geq 0$, given in (2.2.27) are obtained from (2.2.22) taking $M=N=1$.

(iii) (2.2.25) becomes

$$s_{m+n} = \frac{\zeta^2_{0,n+1,\lambda}}{\zeta_{1,n+1,\lambda} - s_{m+n+1}}, \; n\geq 0, \quad c_{1,n,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}} - \frac{\frac{\zeta_{0,n+1,\lambda}}{\zeta_{0,n,\lambda}}}{c_{1,n+1,\lambda}}, \; n\geq 1, \; c_{1,1,\lambda} = \frac{\zeta_{0,1,\lambda}}{\zeta_{1,0,\lambda}}.$$

(iv) From the theory of continued fractions, we can define the sequence $\{\varpi_{n,\lambda}\}_{n\geq 0}$ by

$$\varpi_{0,\lambda} = 1 \quad \text{and} \quad \varpi_{n+1,\lambda} = s_{m+n}\varpi_{n,\lambda}, \quad n\geq 0,$$

and, as a consequence, the first equation in (2.2.25) becomes

$$\varpi_{n+2,\lambda} = \zeta_{1,n+1,\lambda}\varpi_{n+1,\lambda} - \zeta^2_{0,n+1,\lambda}\varpi_{n,\lambda}, \; n\geq 0, \quad \varpi_{1,\lambda} = \zeta_{1,0,\lambda}, \quad \varpi_{0,\lambda} = 1.$$

Thus, since $\zeta_{0,n+1,\lambda}\neq 0$ for $n\geq 0$, from Favard Theorem there exists a sequence of monic polynomials $\{\varpi_n(x;\lambda)\}_{n\geq 0}$ such that $\varpi_n(0;\lambda) = \varpi_{n,\lambda}$, $n\geq 0$, that is orthogonal with respect to some positive definite linear functional because $\zeta_{1,n,\lambda}, \zeta_{0,n+1,\lambda}\in\mathbb{R}$ for $n\geq 0$. Furthermore, since $\varpi_{n,\lambda}\neq 0$ for $n\geq 0$, then (2.2.28) follows.

(v) It is a straightforward consequence of (2.2.16). □

Remark 2.2.12. As in Theorem 2.2.11.*iv*., from the second equation in (2.2.25) we can define recurrently the sequence

$$\theta_{n+1,\lambda} = \frac{\zeta_{0,n+1,\lambda}/\zeta_{0,n,\lambda}}{c_{1,n+1,\lambda}}\theta_{n,\lambda}, \quad n \geq 1, \quad \theta_{1,\lambda} = \frac{\zeta_{0,1,\lambda}}{c_{1,1,\lambda}}\theta_{0,\lambda}, \quad \theta_{0,\lambda} = 1,$$

and, as a consequence, it becomes $\theta_{0,\lambda} = 1$, $\theta_{1,\lambda} = \zeta_{1,0,\lambda}$,

$$\theta_{2,\lambda} = \frac{\zeta_{1,1,\lambda}}{\zeta_{0,1,\lambda}}\theta_{1,\lambda} - \zeta_{0,1,\lambda}\theta_{0,\lambda}, \quad \theta_{n+1,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}}\theta_{n,\lambda} - \frac{\zeta_{0,n,\lambda}}{\zeta_{0,n-1,\lambda}}\theta_{n-1,\lambda}, \quad n \geq 2.$$

Therefore, if $\zeta_{0,n,\lambda} \neq 0$, $n \geq 1$, from Favard Theorem there exists a SMOP $\{\theta_n(x;\lambda)\}_{n\geq 0}$ which satisfies the TTRR

$$\theta_{n+1}(x;\lambda) = \left(x + \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}}\right)\theta_n(x;\lambda) - \frac{\zeta_{0,n,\lambda}}{\zeta_{0,n-1,\lambda}}\theta_{n-1}(x;\lambda), \quad n \geq 2,$$

$$\theta_2(x;\lambda) = \left(x + \frac{\zeta_{1,1,\lambda}}{\zeta_{0,1,\lambda}}\right)\theta_1(x;\lambda) - \zeta_{0,1,\lambda}\theta_0(x;\lambda), \quad \theta_1(x;\lambda) = x + \zeta_{1,0,\lambda}, \quad \theta_0(x;\lambda) = 1,$$

with $\theta_n(0;\lambda) = \theta_{n,\lambda}$, $n \geq 0$, and it is orthogonal with respect to some regular linear functional which is positive definite if $\zeta_{0,n,\lambda} > 0$ for $n \geq 1$. Besides, since $\theta_{n,\lambda} \neq 0$ for $n \geq 0$, then $c_{1,1,\lambda} = \zeta_{0,1,\lambda}\frac{\theta_0(0;\lambda)}{\theta_1(0;\lambda)}$ and

$$c_{1,n+1,\lambda} = \frac{\zeta_{0,n+1,\lambda}}{\zeta_{0,n,\lambda}}\frac{\theta_n(0;\lambda)}{\theta_{n+1}(0;\lambda)}, \quad s_{m+n} = \zeta_{0,n,\lambda}\frac{\theta_{n+1}(0;\lambda)}{\theta_n(0;\lambda)}, \quad n \geq 1.$$

Remark 2.2.13. When (μ_0, μ_1) is a $(1,0)$-coherent pair of order m, the previous Theorem holds taking $b_{1,n} = 0$, $n \geq 0$. Besides, notice that for $n \geq 0$, $\zeta_{0,n,\lambda}$ and $\zeta_{1,n,\lambda}$ become a constant and a linear function of λ, respectively. As a consequence, from (2.2.29) and by induction on n, $\varpi_n(0;\lambda)$ is a polynomial in λ of degree n with leading coefficient $\prod_{j=0}^{n-1}(j+1)_m^2\|Q_j\|_{\mu_1}^2$, for $n \geq 1$. Thus, (2.2.29) reads

$$\widetilde{\varpi}_{n+1}(\lambda) = (\lambda + \alpha_n)\widetilde{\varpi}_n(\lambda) - \beta_n\widetilde{\varpi}_{n-1}(\lambda), \ \ n \geq 0, \quad \widetilde{\varpi}_0(\lambda) = 1, \tag{2.2.30}$$

where

$$\widetilde{\varpi}_n(\lambda) = \frac{\varpi_n(0;\lambda)}{\prod_{j=0}^{n-1}(j+1)_m^2\|Q_j\|_{\mu_1}^2}, \quad n \geq 1,$$

is a monic polynomial and

$$\alpha_0 = \frac{\|P_m\|_{\mu_0}^2}{(1)_m^2\|Q_0\|_{\mu_1}^2}, \quad \beta_0 = 0,$$

$$\alpha_n = \frac{\|P_{n+m}\|_{\mu_0}^2 + \frac{(n+m)^2}{n^2}a_{1,n}^2\|P_{n+m-1}\|_{\mu_0}^2}{(n+1)_m^2\|Q_n\|_{\mu_1}^2}, \quad \beta_n = \frac{a_{1,n}^2\|P_{m+n-1}\|_{\mu_0}^4}{(n)_m^4\|Q_n\|_{\mu_1}^2\|Q_{n-1}\|_{\mu_1}^2}, \quad n \geq 1.$$

Therefore, if $a_{1,n} \neq 0$ for $n \geq 1$, then the Sobolev norms $\{s_n\}_{n\geq 0}$ and the constants $\{c_{1,n,\lambda}\}_{n\geq 0}$ in (2.2.24) satisfy

$$s_{m+n} = \kappa_n \frac{\widetilde{\varpi}_{n+1}(\lambda)}{\widetilde{\varpi}_n(\lambda)}, \; n \geq 0, \quad c_{1,n,\lambda} = \widetilde{\kappa}_n \frac{\widetilde{\varpi}_{n-1}(\lambda)}{\widetilde{\varpi}_n(\lambda)}, \; n \geq 1,$$

with

$$\kappa_n = (n+1)_m^2 \|Q_n\|_{\mu_1}^2, \quad \widetilde{\kappa}_n = a_{1,n} \frac{(n+1)_m}{(n)_m^3} \frac{\|P_{m+n-1}\|_{\mu_0}^2}{\|Q_{n-1}\|_{\mu_1}^2},$$

where $\{\widetilde{\varpi}_n(\lambda)\}_{n\geq 0}$ is a SMOP in λ with respect to some positive definite linear functional, such that the TTRR (2.2.30) holds.

2.2.3 A Numerical Example

Now, we deal with a numerical example in order to illustrate our Algorithm 2.2.8.

Example 2.2.14. Let us consider the Jacobi weight

$$d\mu^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}\chi_{(-1,1)}(x)dx, \quad \alpha, \beta > -1.$$

Let $\{\widehat{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ be its corresponding SMOP. From [105, Example 5.1] and since

$$\frac{\left(\widehat{P}_{n+m}^{(\alpha,\beta)}(x)\right)^{(m)}}{(n+1)_m} = \widehat{P}_n^{(\alpha+m,\beta+m)}(x), \quad n \geq 0,$$

$$\frac{\left(\widehat{P}_{n+3}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n+1)_3} + a_{1,n}\frac{\left(\widehat{P}_{n+2}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n)_3} + a_{2,n}\frac{\left(\widehat{P}_{n+1}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n-1)_3}$$
$$= \widehat{P}_n^{(\alpha-2,\beta)}(x) + b_{1,n}\widehat{P}_{n-1}^{(\alpha-2,\beta)}(x), \quad n \geq 0,$$

holds for $\alpha > 2$ and $\beta > 3$, where

$$b_{1,n} = \frac{2n(n+\alpha-2)}{(2n+\alpha+\beta-3)(2n+\alpha+\beta-2)}, \quad a_{1,n} = -\frac{4n(n+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-3)},$$
$$a_{2,n} = \frac{4n(n-1)(n+\beta-2)(n+\beta-1)}{(2n+\alpha+\beta-4)(2n+\alpha+\beta-3)^2(2n+\alpha+\beta-2)}.$$

Thus, the measures

$$d\mu_0 = d\mu^{\alpha-3,\beta-4} \quad \text{and} \quad d\mu_1 = d\mu^{\alpha-2,\beta}$$

form a $(2,1)$-coherent pair of order 3, with

$$P_n(x) = \widehat{P}_n^{(\alpha-3,\beta-4)} \quad \text{and} \quad Q_n(x) = \widehat{P}_n^{(\alpha-2,\beta)}, \quad \text{for } \alpha > 2, \; \beta > 3.$$

With the help of MAPLE, we applied Algorithm 2.2.8 and Remark 2.2.10 to the function $f : [-1, 1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = e^{-3(x-\frac{1}{10})^2} \sin(10x),$$

in order to compute its Fourier-Sobolev coefficients with respect to the sequence of Sobolev monic polynomials orthogonal with respect to the inner product

$$\langle g(x), h(x)\rangle_\lambda = \int_{-\infty}^{\infty} g(x)h(x)d\mu_0 + \lambda \int_{-\infty}^{\infty} g'''(x)h'''(x)d\mu_1,$$

defined by the $(2, 1)$-coherent pair of order 3,

$$(\mu_0, \mu_1) \equiv (\mu^{1,1}, \mu^{2,5}), \quad \text{i.e.,} \quad (\alpha, \beta) = (4, 5).$$

This is possible because $f \in L^2_{\mu_2}(-1, 1)$ and $f''' \in L^2_{\mu_1}(-1, 1)$.

For the choice $\lambda = 0.1$, Figures 2.2.1, 2.2.2, 2.2.3, and 2.2.4, simultaneously include plots of $f(x)$ and the partial sums of degree 30 of its Fourier-Jacobi and Fourier-Sobolev series, of $f'(x)$ and of the derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$, of $f''(x)$ and of the second derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$, and, of $f'''(x)$ and of the third derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$, respectively, in the intervals $[-1, 1]$, $[0.9, 0.98]$, $[-1, -0.98]$ and $[0.98, 1]$.

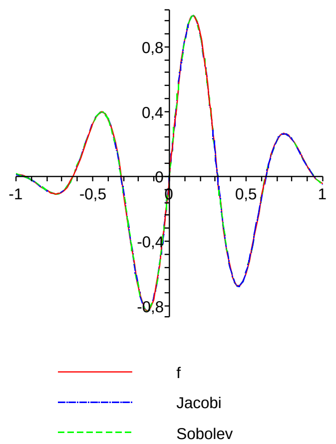
From them, there is a numerical evidence that the approximations for $f(x)$ and its derivatives $f'(x)$, $f''(x)$, $f'''(x)$, given by the partial sums of the Fourier-Sobolev series and its derivatives are better than the corresponding approximations given by the Fourier-Jacobi series and its derivatives. Indeed, Table 2.2.1 illustrates this statement, comparing the errors $\varepsilon^{(i)}_{J,L^2}$, $\epsilon^{(i)}_{J,\mu_0}$, and $E^{(i)}_{J,\lambda}$, with the errors $\varepsilon^{(i)}_{S,L^2}$, $\epsilon^{(i)}_{S,\mu_0}$, and $E^{(i)}_{S,\lambda}$, respectively, given by

$$\begin{aligned}
\varepsilon^{(i)}_{\ell,L^2} &= \|f^{(i)} - S^{(i)}_{30,\ell}(f)\|^2_{L^2} = \int_{-1}^{1} |f^{(i)}(x) - S^{(i)}_{30,\ell}(x; f)|^2 dx, \\
\epsilon^{(i)}_{\ell,\mu_0} &= \|f^{(i)} - S^{(i)}_{30,\ell}(f)\|^2_{\mu_0} = \int_{-1}^{1} \left(f^{(i)}(x) - S^{(i)}_{30,\ell}(x; f)\right)^2 (1 - x^2)dx, \\
E^{(i)}_{\ell,\lambda} &= \|f^{(i)} - S^{(i)}_{30,\ell}(f)\|^2_{\lambda} = \int_{-1}^{1} \left(f^{(i)}(x) - S^{(i)}_{30,\ell}(x; f)\right)^2 (1 - x^2)dx \\
&\qquad + (0.1) \int_{-1}^{1} \left(f^{(3+i)}(x) - S^{(3+i)}_{30,\ell}(x; f)\right)^2 (1 - x)^2(1 + x)^5 dx,
\end{aligned}$$

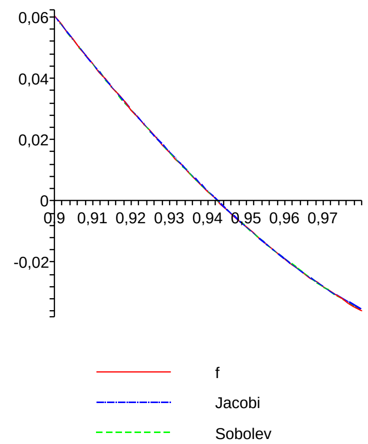
for $i = 0, 1, 2, 3$, and $\ell = J, S$, when approaching the function $f(x)$ $(i = 0)$ and its derivatives $f^{(i)}(x)$, $i = 1, 2, 3$, with the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$ and their derivatives, $S^{(i)}_{30,\ell}(x; f)$, $i = 1, 2, 3$, $\ell = J, S$, for norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{\mu_0}$, and $\|\cdot\|_\lambda$.

Figure 2.2.1: $f(x)$ and the partial sums of degree 30 of its Fourier-Jacobi and Fourier-Sobolev series

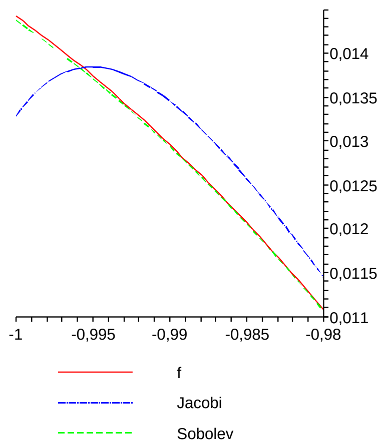
(a) in $[-1, 1]$



(b) in $[0.9, 0.98]$



(c) in $[-1, -0.98]$



(d) in $[0.98, 1]$

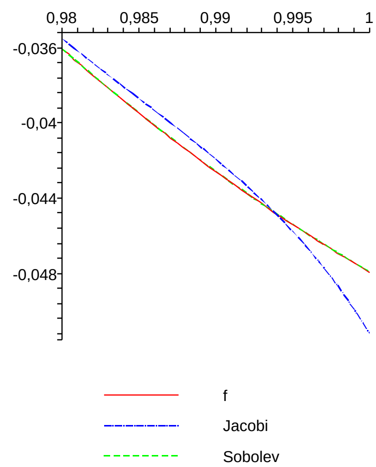


Figure 2.2.2: $f'(x)$ and the derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$

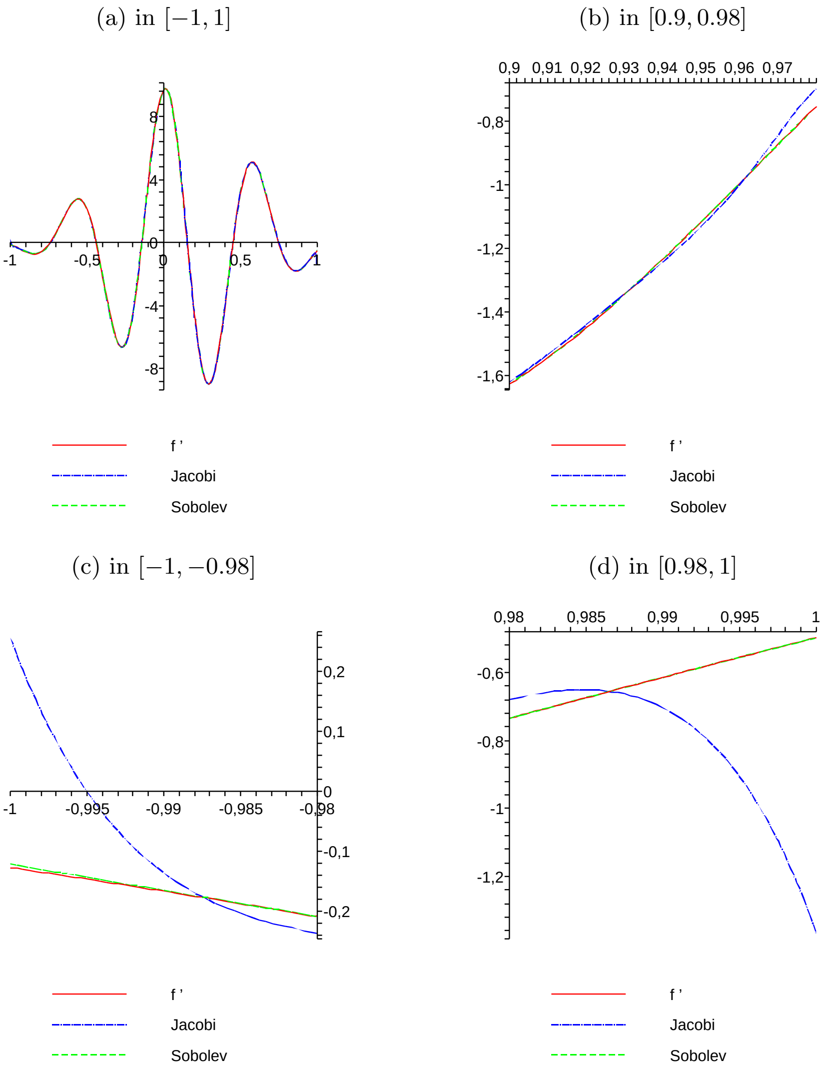
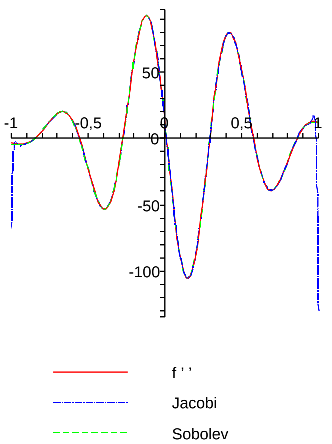
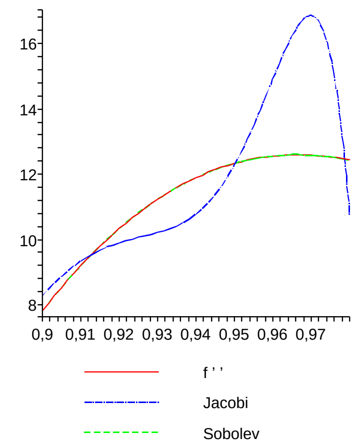


Figure 2.2.3: $f''(x)$ and the second derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$

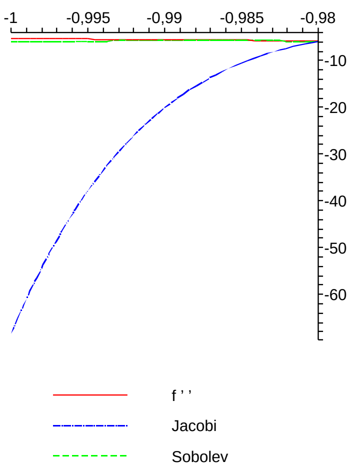
(a) in $[-1,1]$



(b) in $[0.9,0.98]$



(c) in $[-1,-0.98]$



(d) in $[0.98,1]$

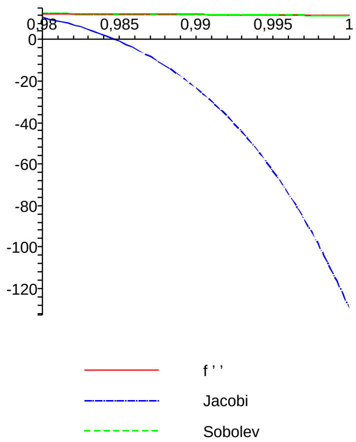
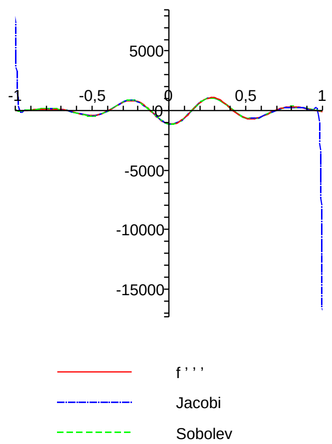
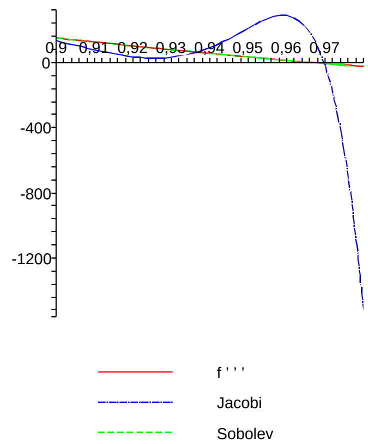


Figure 2.2.4: $f'''(x)$ and the third derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$

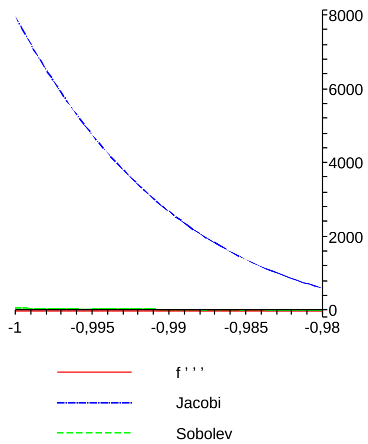
(a) in $[-1,1]$



(b) in $[0.9, 0.98]$



(c) in $[-1,-0.98]$



(d) in $[0.98, 1]$

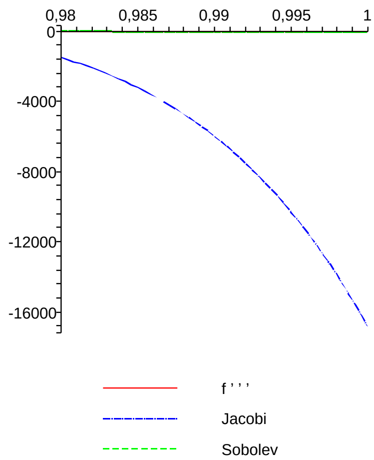


Table 2.2.1: Errors of the approximations of $f(x)$ $(i = 0)$ and its derivatives $(i = 1, 2, 3)$ with the partial sums of degree 30 of the Fourier-Jacobi (J) and Fourier-Sobolev (S) series of $f(x)$ and their derivatives

(a) for the norm $\|\cdot\|_{L^2}$

i	$\varepsilon^{(i)}_{J,L^2}$	$\varepsilon^{(i)}_{S,L^2}$
0	1.05×10^{-7}	1.33×10^{-10}
1	2.76×10^{-3}	2.61×10^{-7}
2	8.99×10	2.35×10^{-3}
3	1.65×10^{6}	2.93×10

(b) for the norm $\|\cdot\|_{\mu_0}$

i	$\epsilon^{(i)}_{J,\mu_0}$	$\epsilon^{(i)}_{S,\mu_0}$
0	1.92×10^{-9}	1.85×10^{-11}
1	1.87×10^{-5}	3.93×10^{-9}
2	6.45×10^{-1}	2.15×10^{-5}
3	1.47×10^{4}	3.05×10^{-1}

(c) for the norm $\|\cdot\|_{\lambda=0.1}$

i	$E^{(i)}_{J,\lambda}$	$E^{(i)}_{S,\lambda}$
0	1.58×10^{2}	4.07×10^{-6}
1	2.54×10^{6}	1.39×10^{-3}
2	2.62×10^{10}	4.30
3	1.82×10^{14}	6.35×10^{4}

2.3 A Matrix Interpretation of (M, N)-Coherence

Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be the SMOP associated with regular linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively. Then, let us consider a relation such as

$$\mathbf{p}'(x) = \mathcal{M}\mathbf{q}(x), \tag{2.3.1}$$

where $\mathcal{M}$ is a infinite matrix and

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{q}(x) = \begin{bmatrix} Q_0(x), & Q_1(x), & \cdots \end{bmatrix}^T.$$

Notice that the entries of the 0th row of $\mathcal{M}$ are all 0's (since $P_0'(x) = 0$). On the other hand, from the TTRR (1.3.2) satisfied by $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, it follows that

$$x\mathbf{p}(x) = \mathcal{J}_P\mathbf{p}(x) \quad \text{and} \quad x\mathbf{q}(x) = \mathcal{J}_Q\mathbf{q}(x) \tag{2.3.2}$$

hold, where $\mathcal{J}_P$ and $\mathcal{J}_Q$ are their corresponding Jacobi Matrices. Thus,

$$\mathcal{M}\mathcal{J}_Q\mathbf{q}(x) + \mathbf{p}(x) \underset{(2.3.2)}{\overset{(2.3.1)}{=}} x\mathbf{p}'(x) + \mathbf{p}(x) = \big(x\mathbf{p}(x)\big)' \overset{(2.3.2)}{=} \mathcal{J}_P\mathbf{p}'(x) \overset{(2.3.1)}{=} \mathcal{J}_P\mathcal{M}\mathbf{q}(x),$$

i.e.,

$$\mathbf{p}(x) = (\mathcal{J}_P\mathcal{M} - \mathcal{M}\mathcal{J}_Q)\,\mathbf{q}(x). \tag{2.3.3}$$

As a consequence,

$$\mathcal{J}_P\left(\mathcal{J}_P\mathcal{M}-\mathcal{M}\mathcal{J}_Q\right)\mathbf{q}(x)\overset{(2.3.3)}{=}\mathcal{J}_P\mathbf{p}(x)\underset{(2.3.2)}{\overset{(2.3.3)}{=}}\left(\mathcal{J}_P\mathcal{M}-\mathcal{M}\mathcal{J}_Q\right)\mathcal{J}_Q\mathbf{q}(x),$$

Therefore, we have proved the following result

Lemma 2.3.1. *If two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are related by an expression such as (2.3.1), i.e.,*

$$\mathbf{p}'(x)=\mathcal{M}\mathbf{q}(x),$$

then

$$\mathcal{J}_P^2\mathcal{M}-2\mathcal{J}_P\mathcal{M}\mathcal{J}_Q+\mathcal{M}\mathcal{J}_Q^2=0, \tag{2.3.4}$$

where $\mathcal{J}_P$ and $\mathcal{J}_Q$ are the Jacobi matrices associated with $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, respectively.

Now, let us consider a (M,N)-coherent pair of regular linear functionals $(\mathcal{U},\mathcal{V})$ given by

$$\frac{P'_{n+1}(x)}{n+1}+\sum_{i=1}^{M}a_{i,n}\frac{P'_{n-i+1}(x)}{n-i+1}=Q_n(x)+\sum_{i=1}^{N}b_{i,n}Q_{n-i}(x),\quad n\geq 0, \tag{2.3.5}$$

where $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the SMOP associated with $\mathcal{U}$ and $\mathcal{V}$, respectively, $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, and $a_{i,n}=b_{i,n}=0$ if $i>n$. In order to use Lemma 2.3.1, the (M,N)-coherence relation (2.3.5) can be written in two matrix forms as

$$\mathcal{A}_1\mathbf{p}_1'(x)=\mathcal{B}\mathbf{q}(x), \tag{2.3.6}$$
$$\mathcal{A}\mathbf{p}'(x)=\mathcal{B}\mathbf{q}(x), \tag{2.3.7}$$

where $\mathbf{p}(x)$ and $\mathbf{q}(x)$ are given as in (2.3.1),

$$\mathbf{p}_1(x)=\begin{bmatrix} P_1(x), & P_2(x), & \cdots \end{bmatrix}^T,$$

$\mathcal{A}$ is a lower Hessenberg matrix with $M+1$ nonzero diagonals (such that the entries of its superdiagonal are $\frac{1}{n+1}$, $n\geq 0$, and the entries of its main diagonal are 1 and $\frac{a_{1,n}}{n}$, $n\geq 1$), and, $\mathcal{A}_1$ and $\mathcal{B}$ are nonsingular lower triangular matrices with $M+1$ and $N+1$ nonzero diagonals, respectively, (whose entries of their main diagonals are $\frac{1}{n+1}$, $n\geq 0$, and 1's, respectively). These infinite matrices $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{B}$ are such that

$$\left[\begin{array}{cccccccc} \underbrace{0 \ \cdots \ 0}_{n-M+1 \text{ zeros}} & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{2,n}}{n-1} & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots \end{array}\right],$$

$\uparrow$
nth position
$\downarrow$

$$\left[\begin{array}{ccccccc} \underbrace{0 \ \cdots \ 0}_{n-M \text{ zeros}} & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots \end{array}\right],$$

$\uparrow$
nth position
$\downarrow$

$$\left[\begin{array}{ccccccc} \underbrace{0 \ \cdots \ 0}_{n-N \text{ zeros}} & b_{N,n} & \cdots & b_{1,n} & 1 & 0 & \cdots \end{array}\right]$$

are their corresponding nth rows, for $n \geq 1$, and, $[1\ 1\ 0\ \cdots]$, $[1\ 0\ \cdots]$, and $[1\ 0\ \cdots]$ are their 0th rows, respectively, (counting the rows from zero).

Proposition 2.3.2. *Let $(\mathcal{U}, \mathcal{V})$ be a (M, N)-coherent pair given by (2.3.6) and let $\mathcal{J}_P$ and $\mathcal{J}_Q$ be the Jacobi matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Then*

$$\mathcal{J}_P^2 \left[\begin{array}{c} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{array}\right] - 2\mathcal{J}_P \left[\begin{array}{c} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{array}\right] \mathcal{J}_Q + \left[\begin{array}{c} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{array}\right] \mathcal{J}_Q^2 = 0$$

holds, where $\mathbf{0}$ is the zero row, i.e., $\mathbf{0} = \left[\ 0, \quad 0, \quad \cdots\ \right]$.

Proof. (2.3.6) can be read as (2.3.1) as follows

$$\mathbf{p}_1'(x) = \mathcal{A}_1^{-1}\mathcal{B}\mathbf{q}(x) \quad \Longrightarrow \quad \mathbf{p}'(x) = \left[\begin{array}{c} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{array}\right] \mathbf{q}(x),$$

i.e., the matrix $\mathcal{M}$ in (2.3.1) is the matrix obtained from $\mathcal{A}_1^{-1}\mathcal{B}$ by shifting the matrix one position downward, adding a zero row to top. As a consequence, Lemma 2.3.1 holds and, in particular, (2.3.4) holds replacing $\mathcal{M}$ by $\left[\begin{array}{c} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{array}\right]$. $\square$

Proposition 2.3.3. *If $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair given by (2.3.7) such that $\mathcal{A}$ is a nonsingular matrix (for example, when $M = 1$ and $N \geq 0$, $\mathcal{A}$ is a nonsingular upper bidiagonal matrix, since $a_{1,n} \neq 0$ for $n \geq 1$), then*

$$(\mathcal{M}_P - \mathcal{M}_Q)^2 = [\mathcal{M}_P, \mathcal{M}_Q],$$

where $[\mathcal{M}_P, \mathcal{M}_Q]$ is the commutator of the matrices $\mathcal{M}_P$ and $\mathcal{M}_Q$ defined by

$$[\mathcal{M}_P, \mathcal{M}_Q] = \mathcal{M}_P\mathcal{M}_Q - \mathcal{M}_Q\mathcal{M}_P,$$

and, $\mathcal{M}_P$ and $\mathcal{M}_Q$ are given by

$$\mathcal{M}_P = \mathcal{A}\mathcal{J}_P\mathcal{A}^{-1} \quad \text{and} \quad \mathcal{M}_Q = \mathcal{B}\mathcal{J}_Q\mathcal{B}^{-1},$$

i.e., $\mathcal{M}_P$ and $\mathcal{M}_Q$ are similar matrices to the monic Jacobi matrices $\mathcal{J}_P$ and $\mathcal{J}_Q$ associated with $\mathcal{U}$ and $\mathcal{V}$, respectively.

Proof. Let $\mathcal{M} = \mathcal{A}^{-1}\mathcal{B}$. Then, from (2.3.7) and Lemma 2.3.1,

$$\mathcal{J}_P^2\mathcal{A}^{-1}\mathcal{B} - 2\mathcal{J}_P\mathcal{A}^{-1}\mathcal{B}\mathcal{J}_Q + \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_Q^2 = 0$$

holds. Therefore, if we multiply in the left by $\mathcal{A}$ and in the right by $\mathcal{B}^{-1}$ in both sides of the previous equation, we obtain

$$0 = \mathcal{M}_P^2 - 2\mathcal{M}_P\mathcal{M}_Q + \mathcal{M}_Q^2 = (\mathcal{M}_P - \mathcal{M}_Q)^2 - [\mathcal{M}_P, \mathcal{M}_Q],$$

which is the desired result. □

When $\mathcal{U}$ is a classical linear functional we get the following additional characterization.

Proposition 2.3.4. *Let $(\mathcal{U}, \mathcal{V})$ be a (M, N)-coherent pair of order m, $m \geq 0$, of regular linear functionals given by*

$$\widehat{\mathcal{A}}_1\widehat{\mathbf{p}}(x) = \mathcal{B}\mathbf{q}(x), \tag{2.3.8}$$

where $\mathcal{B}$ and $\mathbf{q}(x)$ are given as in (2.3.6),

$$\widehat{\mathbf{p}}(x) = \begin{bmatrix} P_0^{[m]}(x), & P_1^{[m]}(x), & \cdots \end{bmatrix}^T,$$

and $\widehat{\mathcal{A}}_1$ is the lower triangular matrix with $M+1$ nonzero diagonals, such that its nth row for $n \geq 1$ (counting the rows from zero), is

$$[\quad \underbrace{0 \quad \cdots \quad 0}_{n-M \ \text{zeros}} \quad a_{M,n} \quad \cdots \quad a_{1,n} \quad 1 \quad 0 \quad \cdots \quad],$$

$a_{M,n} \neq 0$, $n \geq M$, $a_{i,n} = 0$, $i > n$, and the entries of its main diagonal are all 1's.

If $\mathcal{U}$ is a classical linear functional, then $\mathcal{J}_{P^{[m]}}$ and $\mathcal{J}_Q$, the monic Jacobi matrices associated with the SMOP $\{P_n^{[m]}(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ respectively, are similar matrices satisfying

$$\widehat{\mathcal{A}}_1\mathcal{J}_{P^{[m]}}\widehat{\mathcal{A}}_1^{-1} = \mathcal{M}_{P^{[m]}} = \mathcal{M}_Q = \mathcal{B}\mathcal{J}_Q\mathcal{B}^{-1}.$$

Proof. From Theorem 1.5.1, $\{P_n^{[m]}(x)\}_{n\geq 0}$ is also a classical SMOP. Hence, from (1.3.2), we get

$$x\widehat{\mathbf{p}}(x) = \mathcal{J}_{P^{[m]}}\widehat{\mathbf{p}}(x).$$

Thus, multiplying the previous equation by $\widehat{\mathcal{A}}_1$ and taking into account that $\widehat{\mathcal{A}}_1$ and $\mathcal{B}$ are nonsingular matrices, we get

$$\widehat{\mathcal{A}}_1\mathcal{J}_{P^{[m]}}\widehat{\mathcal{A}}_1^{-1}\mathcal{B}\mathbf{q}(x) \stackrel{(2.3.8)}{=} \widehat{\mathcal{A}}_1\mathcal{J}_{P^{[m]}}\widehat{\mathbf{p}}(x) = x\widehat{\mathcal{A}}_1\widehat{\mathbf{p}}(x) \stackrel{(2.3.8)}{=} x\mathcal{B}\mathbf{q}(x) \stackrel{(2.3.2)}{=} \mathcal{B}\mathcal{J}_Q\mathbf{q}(x).$$

Since the basis are the same, the proof is complete. □

Remark 2.3.5. Proposition 2.3.4 can be used for Hermite, Laguerre, and Jacobi SMOP taking into account the information presented in Table 1.5.1.

Remark 2.3.6. The well known Hahn's condition characterizing classical orthogonal polynomials (see [42]) has been considered from a matrix approach in [114] where nice and elegant proofs of some well known characterizations of classical orthogonal polynomials are given. Therein, the connection between the Jacobi matrices associated with the SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{\frac{P_n'(x)}{n+1}\}_{n\geq 0}$ is stated in formula (4.2) and, as a consequence, it is possible to get the general expression of the coefficients of the TTRR for classical orthogonal polynomials by identifying the corresponding entries. Notice that the statement of Lemma 2.3.1 generalizes (4.2) for families of orthogonal polynomials related by a relation like $\mathbf{p}'(x) = \mathcal{M}\mathbf{q}(x)$. On the other hand, (4.2) is a straightforward consequence of Proposition 2.3.2 when $\mathcal{A}_1 = diag(1, 1/2, 1/3, ...)$ and $\mathcal{B} = I$.

2.3.1 A Matrix Interpretation of Sobolev Orthogonal Polynomials and (M,N)-Coherence of Order m

Let us consider the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$ analyzed in Section 2.2, which is orthogonal with respect the inner product (2.2.1), this is,

$$\langle p(x), r(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)r(x)d\mu_0 + \lambda\int_{\mathbb{R}} p^{(m)}(x)r^{(m)}(x)d\mu_1, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients and μ_0 and μ_1 are positive Borel measures supported on an infinite subset of the real line.

In Theorem 2.2.2, we proved that if $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order m, $m\geq 1$, of positive definite linear functionals such that μ_0 and μ_1 are their corresponding positive Borel measures, then the SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{S_n(x;\lambda)\}_{n\geq 0}$ with respect to $\mathcal{U}$ and $\langle\cdot,\cdot\rangle_\lambda$, respectively, satisfy the algebraic relation (2.2.8). So, using the notation given in (2.2.15)

$$\widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n}, \quad n\geq 0,$$

where $\widetilde{a}_{i,n} = 0$ when $i > n$, and $\widetilde{a}_{0,n} = 1$, $n\geq 0$, (2.2.8) reads

$$P_{n+m}(x) + \sum_{i=1}^{M}\widetilde{a}_{i,n}P_{n-i+m}(x) = S_{n+m}(x;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda}S_{n-j+m}(x;\lambda), \quad n\geq 0,$$

$$S_n(x;\lambda) = P_n(x), \quad n \leq m,$$

where $c_{j,n,\lambda} = 0$, $n < j \leq K$, and $K = \max\{M, N\}$. Hence, we can express these relations as

$$\widetilde{\mathcal{A}}\mathbf{p}(x) = \mathcal{C}\mathbf{s}(x;\lambda), \tag{2.3.9}$$

where

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{s}(x;\lambda) = \begin{bmatrix} S_0(x;\lambda), & S_1(x;\lambda), & \cdots \end{bmatrix}^T,$$

and the matrices $\widetilde{\mathcal{A}}$ and $\mathcal{C}$ are banded matrices. More precisely, they are lower triangular matrices with $M+1$ and $K+1$ nonzero diagonals, respectively, whose entries of their main diagonal are all 1's, their first m rows (counting from zero) are $[0\ \cdots\ 0\ 1\ 0\ \cdots]$, and their $(n+m)$th rows, $n \geq 0$, are, respectively,

$$\left[\begin{array}{ccccccc} \underbrace{0\ \ \cdots\ \ 0}_{n-M+m \text{ zeros}} & \widetilde{a}_{M,n} & \cdots & \widetilde{a}_{1,n} & 1 & 0 & \cdots \end{array}\right],$$

$$\begin{array}{c} \uparrow \\ (n+m)\text{th place} \\ \downarrow \end{array}$$

$$\left[\begin{array}{ccccccc} \underbrace{0\ \ \cdots\ \ 0}_{n-K+m \text{ zeros}} & c_{K,n,\lambda} & \cdots & c_{1,n,\lambda} & 1 & 0 & \cdots \end{array}\right].$$

Therefore,

$$x\mathbf{s}(x;\lambda) \overset{(2.3.9)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}x\mathbf{p}(x) \overset{(1.3.2)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{J}_P\mathbf{p}(x) \overset{(2.3.9)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{J}_P\widetilde{\mathcal{A}}^{-1}\mathcal{C}\mathbf{s}(x;\lambda).$$

In other words, the matrix representation of the multiplication operator by x in terms of the basis $\{S_n(x;\lambda)\}_{n\geq 0}$ is

$$x\mathbf{s}(x;\lambda) = \mathcal{H}_{S,P,\lambda}\mathbf{s}(x;\lambda),$$

where $\mathcal{H}_{S,P,\lambda}$ is the infinite lower Hessenberg matrix

$$\mathcal{H}_{S,P,\lambda} = \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{J}_P\widetilde{\mathcal{A}}^{-1}\mathcal{C},$$

similar to the monic Jacobi matrix $\mathcal{J}_P$ associated with the SMOP $\{P_n(x)\}_{n\geq 0}$.

CHAPTER 3

D_ν-Coherent Pairs, for $\nu = \omega, q$

Let ν be either $\omega \in \mathbb{C} \setminus \{0\}$ or $q \in \mathbb{C} \setminus \{0, 1\}$. A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a *(M, N)-D_ν-coherent pair of order (m, k)* if their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$\sum_{i=0}^{M} a_{i,n} D_\nu^m P_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} D_\nu^k Q_{n+k-i}(x), \quad n \geq 0,$$

where $D_\omega p(x) = \frac{p(x+\omega)-p(x)}{\omega}$, $D_q p(x) = \frac{p(qx)-p(x)}{(q-1)x}$, $M, N, m, k \in \mathbb{N} \cup \{0\}$, $a_{i,n}, b_{i,n} \in \mathbb{C}$, for $n \geq 0$, $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ for $i > n$. Moreover, $(\mathcal{U}, \mathcal{V})$ is called a *(M, N)-D_ν-coherent pair of order m* if it is a (M, N)-D_ν-coherent pair of order $(m, 0)$, and if also $m = 1$, $(\mathcal{U}, \mathcal{V})$ is said to be a *(M, N)-D_ν-coherent pair*. On the other hand, when $\mathcal{U}$ and $\mathcal{V}$ are weakly quasi-definite linear functionals of order Υ_0 and Υ_1, respectively, all the previous definitions hold with the additional restrictions $0 \leq n \leq \min\{\Upsilon_0 - m, \Upsilon_1 - k\}$, $0 \leq M, m \leq \Upsilon_0$, and $0 \leq N, k \leq \Upsilon_1$.

The structure of this chapter is as follows. In Section 3.1, we will prove that if a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-D_ν-coherent pair, then at least one of them must be D_ν-semiclassical of class at most 1, and these functionals are related by $\tilde{\sigma}(x)\mathcal{U} = \rho(x)\mathcal{V}$, where $\tilde{\sigma}(x)$ and $\rho(x)$ are polynomials of degrees ≤ 3 and 1, respectively, so the other linear functional is also D_ν-semiclassical of class at most 5. Besides, the D_ν-classical case will be studied. Finally, the case when $\mathcal{U}$ and $\mathcal{V}$ are weakly quasi-definite linear functionals is analyzed. In Section 3.2, we will prove that (M, N)-D_ν-coherence of order (m, k) implies that the regular linear functionals are D_ν-semiclassical, whenever $m \neq k$, and they are related by a rational factor. Conversely, the D_ν-semiclassical character and a relation of rational type for a pair of regular linear functionals yield (M,N)-D_ν-coherence. In Section 3.3, we will show that there is a close relation between (M,N)-D_ν-coherent pairs of order m and D_ν-Sobolev orthogonal polynomials associated with the inner product

$$\langle p(x), r(x)\rangle_{\lambda,\nu} = \langle \mathcal{U}, p(x)r(x)\rangle + \lambda \langle \mathcal{V}, (D_\nu^m p)(x)(D_\nu^m r)(x)\rangle, \quad \lambda > 0,$$

for fixed $m \geq 1$. In this framework, $(1,0)$ and $(1,1)$ -D_ν-coherent pairs of order m are two special cases which will be analyzed. Finally, in Section 3.4, we will present a matrix interpretation of (M,N)-D_ν-coherence for a pair of regular linear functionals in terms of the corresponding monic Jacobi matrices. In this way, we also will study (M,N)-D_ν-coherent pairs of order m such that one of the linear functionals is D_ν-classical. On the other hand, we will obtain a matrix representation of the multiplication operator by x in terms of the bases of the D_ν-Sobolev orthogonal polynomials studied in Section 3.3.

3.1 $(1,1)$-D_ν-Coherent Pairs

Let us consider a $(1,1)$-D_ν*-coherent pair* of regular linear functionals $(\mathcal{U},\mathcal{V})$ defined by

$$P_n^{[1,\nu]}(x) + a_n P_{n-1}^{[1,\nu]}(x) = Q_n(x) + b_n Q_{n-1}(x), \quad a_n \neq 0,\ n \geq 1, \tag{3.1.1}$$

where $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the corresponding SMOP of $\mathcal{U}$ and $\mathcal{V}$. Remember that $(\mathcal{U},\mathcal{V})$ is called a $(1,0)$-D_ν*-coherent pair* when $b_n = 0$ for all $n \geq 1$. We will also say that the pair $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$ is a $(1,1)$-D_ν*-coherent pair* or $(1,0)$-D_ν*-coherent pair* of SMOP, respectively.

For the case when $\mathcal{U}$ and $\mathcal{V}$ are weakly quasi-definite linear functionals, to the end of this section, we will present as a remark (Remark 3.1.19) the corresponding results obtained through this section.

The aim of the first part of this section is to prove Theorems 3.1.7, 3.1.10 and 3.1.12 which state the D_ν-analogue results of A. Delgado and F. Marcellán ([33]) for $(1,1)$-coherent pairs, and generalize the results obtained by I. Area, E. Godoy, and F. Marcellán ([14, 15, 16, 17]) for $(1,0)$-D_ν-coherent pairs.

Remark 3.1.1. If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ν-coherent pair given by (3.1.1) then

- $a_1 \neq b_1$ if and only if $P_n^{[1,\nu]}(x) \neq Q_n(x)$ for all $n \geq 1$.
- For $n \geq 1$,

$$\begin{aligned} P_n^{[1,\nu]}(x) &= Q_n(x) + (b_n - a_n)Q_{n-1}(x) \\ &\quad + \sum_{k=2}^{n} (-1)^{k-1} a_n a_{n-1}\cdots a_{n-(k-2)} \left(b_{n-(k-1)} - a_{n-(k-1)}\right) Q_{n-k}(x). \end{aligned} \tag{3.1.2}$$

Proof. For the first statement, since $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are monic polynomials, using (3.1.1) it is easy to prove that there exists $N\geq 1$ such that $P_N^{[1,\nu]}(x)=Q_N(x)$ if and only if $a_1=b_1$. On the other hand, the proof of (3.1.2) can be done by induction on n. □

Lemma 3.1.2. *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1\neq b_1$, then there exists a monic polynomial $\gamma_n(x;\nu)$ of degree $n\geq 1$ such that*

$$\left\langle \gamma_n(x;\nu)\mathcal{V}, P_m^{[1,\nu]}(x)\right\rangle = 0, \quad \textit{for } m\geq n+1. \tag{3.1.3}$$

Proof. If $\gamma_n(x;\nu)=Q_n(x)+\sum_{j=0}^{n-1}A_{j,n,\nu}Q_j(x)$, then from (3.1.2), we obtain, for $n\geq 0$,

$$\begin{aligned}\left\langle \gamma_n(x;\nu)\mathcal{V}, P_{n+1}^{[1,\nu]}(x)\right\rangle &= \left\langle \mathcal{V}, \gamma_n(x;\nu)P_{n+1}^{[1,\nu]}(x)\right\rangle \\ &= (b_{n+1}-a_{n+1})\langle \mathcal{V}, Q_n^2(x)\rangle + \sum_{k=2}^{n+1}(-1)^{k-1}a_{n+1}\cdots a_{n+1-(k-2)} \\ &\qquad \left(b_{n+1-(k-1)}-a_{n+1-(k-1)}\right) A_{n+1-k,n,\nu}\langle \mathcal{V}, Q_{n+1-k}^2(x)\rangle.\end{aligned}$$

Since $a_1\neq b_1$, we can choose $A_{0,n,\nu},\ldots,A_{n-1,n,\nu}$, not all zero, such that

$$\langle \gamma_n(x;\nu)\mathcal{V}, P_{n+1}^{[1,\nu]}(x)\rangle = 0, \quad n\geq 1.$$

On the other hand, if we apply $\langle \gamma_n(x;\nu)\mathcal{V},\cdot\rangle$ to (3.1.1), we get

$$\langle \gamma_n(x;\nu)\mathcal{V}, P_{m+1}^{[1,\nu]}(x)\rangle = -a_{m+1}\langle \gamma_n(x;\nu)\mathcal{V}, P_m^{[1,\nu]}(x)\rangle, \quad n\leq m-1.$$

Therefore, the lemma follows. □

Remark 3.1.3. In Lemma 3.1.2, we can choose $A_{1,n,\nu}=\cdots=A_{n-1,n,\nu}=0$. Thus,

$$\gamma_n(x;\nu)=Q_n(x)+A_{0,n,\nu}=Q_n(x)+\frac{(-1)^{n+1}(b_{n+1}-a_{n+1})\langle \mathcal{V}, Q_n^2(x)\rangle}{a_{n+1}a_n\cdots a_3a_2(b_1-a_1)\langle \mathcal{V},1\rangle}, \quad n\geq 1. \tag{3.1.4}$$

Lemma 3.1.4. *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1\neq b_1$, then*

$$D_{\nu^*}[\gamma_n(x;\nu)\mathcal{V}] = -\varphi_{n+1}(x;\nu)\mathcal{U}, \quad n\geq 1, \tag{3.1.5}$$

where $\gamma_n(x;\nu)$ is the monic polynomial of degree n introduced in Lemma 3.1.2 and $\varphi_{n+1}(x;\nu)$ is a polynomial of degree at most $n+1$. Moreover,

$$\varphi_{n+1}(x;\nu)=\sum_{k=0}^{n}\frac{\eta_{k,1,\nu}\langle \gamma_n(x;\nu)\mathcal{V}, P_k^{[1,\nu]}(x)\rangle}{\langle \mathcal{U}, P_{k+1}^2(x)\rangle}P_{k+1}(x), \quad n\geq 1. \tag{3.1.6}$$

Proof. Let $\{\mathfrak{p}_n\}_{n\geq 0}$ and $\{\mathfrak{e}_{n,1,\nu}\}_{n\geq 0}$ be the dual bases of $\{P_n(x)\}_{n\geq 0}$ and $\{P_n^{[1,\nu]}(x)\}_{n\geq 0}$, respectively. Since

$$\gamma_n(x;\nu)\mathcal{V} = \sum_{k\geq 0} \lambda_{k,n,\nu}\mathfrak{e}_{k,1,\nu} \quad \text{where} \quad \lambda_{k,n,\nu} = \left\langle \gamma_n(x;\nu)\mathcal{V}, P_k^{[1,\nu]}(x) \right\rangle,$$

and from Lemma 3.1.2, $\lambda_{k,n,\nu} = 0$ for $k \geq n+1$ and $n \geq 1$, then

$$\gamma_n(x;\nu)\mathcal{V} = \sum_{k=0}^{n} \lambda_{k,n,\nu}\mathfrak{e}_{k,1,\nu}.$$

Thus, from (1.3.5) it follows

$$D_{\nu^*}\left[\gamma_n(x;\nu)\mathcal{V}\right] = \sum_{k=0}^{n} \lambda_{k,n,\nu}\left(-\eta_{k,1,\nu}\mathfrak{p}_{k+1}\right) = -\sum_{k=0}^{n} \lambda_{k,n,\nu}\eta_{k,1,\nu}\frac{P_{k+1}(x)}{\langle \mathcal{U}, P_{k+1}^2(x)\rangle}\mathcal{U}, \quad n \geq 1.$$

□

Corollary 3.1.5. *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1 \neq b_1$, then there exist polynomials $\alpha(x;\nu)$, $\beta(x;\nu)$, and $\phi(x;\nu)$ such that*

$$\alpha(x;\nu)\mathcal{U} = \beta(x;\nu)\mathcal{V}, \tag{3.1.7}$$

$$\alpha(x;\nu)D_{\nu^*}\mathcal{V} = \phi(x;\nu)\mathcal{V}, \tag{3.1.8}$$

$$\phi(x;\nu)\mathcal{U} = \beta(x;\nu)D_{\nu^*}\mathcal{V}, \tag{3.1.9}$$

where

$$\alpha(x;\nu) = \hbar_\nu\left[\gamma_2(x\star\nu^*;\nu)\varphi_2(x;\nu) - \gamma_1(x\star\nu^*;\nu)\varphi_3(x;\nu)\right], \tag{3.1.10}$$

$$\beta(x;\nu) = \gamma_1(x\star\nu^*;\nu)(D_{\nu^*}\gamma_2)(x;\nu) - \gamma_2(x\star\nu^*;\nu), \tag{3.1.11}$$

$$\phi(x;\nu) = \varphi_3(x;\nu) - (D_{\nu^*}\gamma_2)(x;\nu)\varphi_2(x;\nu), \tag{3.1.12}$$

with $\deg\left(\alpha(x;\nu)\right) \leq 4, \deg\left(\beta(x;\nu)\right) = 2, \deg\left(\phi(x;\nu)\right) \leq 3$. *Furthermore,*

$$\phi(x;\nu)\gamma_n(x\star\nu^*;\nu) + \hbar_{\nu^*}\alpha(x;\nu)\left(D_{\nu^*}\gamma_n\right)(x;\nu) = -\varphi_{n+1}(x;\nu)\beta(x;\nu), \quad n \geq 1, \tag{3.1.13}$$

where $\gamma_n(x;\nu)$ and $\varphi_{n+1}(x;\nu)$ are the polynomials given in Lemma 3.1.4.

Proof. From (3.1.5) for $n = 1$ and $n = 2$, we get

$$\gamma_1(x\star\nu^*;\nu)D_{\nu^*}\mathcal{V} + \hbar_{\nu^*}\mathcal{V} = -\varphi_2(x;\nu)\mathcal{U}, \tag{3.1.14}$$

$$\gamma_2(x\star\nu^*;\nu)D_{\nu^*}\mathcal{V} + \hbar_{\nu^*}(D_{\nu^*}\gamma_2)(x;\nu)\mathcal{V} = -\varphi_3(x;\nu)\mathcal{U}. \tag{3.1.15}$$

Then, by elimination of $D_\nu\mathcal{V}, \mathcal{U}$, and $\mathcal{V}$ we obtain (3.1.7), (3.1.8) and (3.1.9), respectively. Besides, the degrees of $\alpha(x;\nu)$ and $\phi(x;\nu)$ follows, and $\deg(\beta(x;\nu)) = 2$ since its leading coefficient is $\hbar_{\nu^*}\eta_{1,1,\nu^*} - \hbar^2_{\nu^*} = \hbar_{\nu^*} \neq 0$. Finally,

$$\begin{aligned}-\varphi_{n+1}(x;\nu)\beta(x;\nu)\mathcal{V} &\overset{(3.1.7)}{=} -\alpha(x;\nu)\varphi_{n+1}(x;\nu)\mathcal{U} \overset{(3.1.5)}{=} \alpha(x;\nu)D_{\nu^*}[\gamma_n(x;\nu)\mathcal{V}] \\ &\overset{(3.1.8)}{=} [\gamma_n(x\star\nu^*;\nu)\phi(x;\nu) + \hbar_{\nu^*}\alpha(x;\nu)\left(D_{\nu^*}\gamma_n\right)(x;\nu)]\,\mathcal{V}, \quad n\geq 1.\end{aligned}$$

□

Remark 3.1.6. From (3.1.6), the leading coefficients of $\varphi_2(x;\nu)$ and $\varphi_3(x;\nu)$ are, respectively,

$$\eta_{1,1,\nu}\frac{\left\langle \mathcal{V}, \gamma_1(x;\nu)P_1^{[1,\nu]}(x)\right\rangle}{\langle \mathcal{U}, P_2^2(x)\rangle} \underset{(3.1.2)}{\overset{(3.1.4)}{=}} \frac{\eta_{1,1,\nu}\left\langle \mathcal{V}, Q_1^2(x)\right\rangle}{a_2\langle \mathcal{U}, P_2^2(x)\rangle}b_2,$$

$$\eta_{2,1,\nu}\frac{\langle \gamma_2(x;\nu)\mathcal{V}, P_2^{[1,\nu]}(x)\rangle}{\langle \mathcal{U}, P_3^2(x)\rangle} \underset{(3.1.2)}{\overset{(3.1.4)}{=}} \frac{\eta_{2,1,\nu}\langle \mathcal{V}, Q_2^2(x)\rangle}{a_3\langle \mathcal{U}, P_3^2(x)\rangle}b_3.$$

As a consequence, from (3.1.10), (3.1.11) and (3.1.12), the leading coefficients of $\beta(x;\nu)$, $\alpha(x;\nu)$, and $\phi(x;\nu)$ are, respectively, $\hbar_{\nu^*}$,

$$\hbar_{\nu^*}\frac{\eta_{1,1,\nu}\left\langle \mathcal{V}, Q_1^2(x)\right\rangle}{a_2\langle \mathcal{U}, P_2^2(x)\rangle}b_2 - \frac{\eta_{2,1,\nu}\langle \mathcal{V}, Q_2^2(x)\rangle}{a_3\langle \mathcal{U}, P_3^2(x)\rangle}b_3,$$

$$\frac{\eta_{2,1,\nu}\langle \mathcal{V}, Q_2^2(x)\rangle}{a_3\langle \mathcal{U}, P_3^2(x)\rangle}b_3 - \eta_{1,1,\nu^*}\frac{\eta_{1,1,\nu}\left\langle \mathcal{V}, Q_1^2(x)\right\rangle}{a_2\langle \mathcal{U}, P_2^2(x)\rangle}b_2.$$

To prove that $(1,1)$-D_ν-coherence for $\mathcal{U}$ and $\mathcal{V}$ yields they are D_ν-semiclassical linear functionals, we consider the zeros of the monic polynomial $\beta(x;\nu)$ given by (3.1.11). Notice that

$$\begin{aligned}&\left(D_\nu\beta\right)(x;\nu) \\ &\quad= D_\nu\left[\gamma_1(x\star\nu^*;\nu)\right]\left(D_{\nu^*}\gamma_2\right)(x;\nu) + \gamma_1(x;\nu)D_\nu\left[\left(D_{\nu^*}\gamma_2\right)(x;\nu)\right] - D_\nu\left[\gamma_2(x\star\nu^*;\nu)\right] \\ &\quad\quad= \hbar_{\nu^*}(D_\nu\gamma_1)(x\star\nu^*;\nu)\left(D_{\nu^*}\gamma_2\right)(x;\nu) + \gamma_1(x;\nu)\eta_{1,1,\nu^*} - \hbar_{\nu^*}(D_\nu\gamma_2)(x\star\nu^*;\nu) \\ &\quad\quad\quad= \hbar_{\nu^*}\left(D_{\nu^*}\gamma_2\right)(x;\nu) + \hbar_{\nu^*}\eta_{1,1,\nu}\gamma_1(x;\nu) - \hbar_{\nu^*}\left(D_{\nu^*}\gamma_2\right)(x;\nu) = \eta_{1,1,\nu^*}\gamma_1(x;\nu),\end{aligned}$$

and if $\xi_{1,\nu}$ and $\xi_{2,\nu}$ denote the zeros of $\beta(x;\nu)$, then

$$\beta(x;\nu) = \hbar_{\nu^*}(x-\xi_{1,\nu})(x-\xi_{2,\nu}), \quad \left(D_\nu\beta\right)(x;\nu) = \hbar_{\nu^*}\eta_{1,1,\nu}\gamma_1(x;\nu),$$
$$\left(D_\omega\beta\right)(x;\omega) = 2\left[x - \frac{\xi_{1,\nu}+\xi_{2,\nu}-\omega}{2}\right], \quad \left(D_q\beta\right)(x;q) = q^{-1}[2]_q\left[x - \frac{\xi_{1,\nu}+\xi_{2,\nu}}{q+1}\right]. \tag{3.1.16}$$

Therefore, the cases to analyze are the following:

i. ξ_ν and $\xi_\nu \star \nu^*$ are the zeros of $\beta(x;\nu)$, equivalently, ξ_ν is a zero of $\beta(x;\nu)$ such that $\xi_\nu \star \nu^*$ is the zero of $(D_\nu \beta)(x;\nu)$. In this case, (see Theorem 3.1.7) we will conclude that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ν-coherent pair of regular linear functionals, then $\mathcal{U}$ and $\mathcal{V}$ are D_ν-semiclassical of class at most 1 and 5 respectively.

ii. $\xi_{1,\nu}$ and $\xi_{2,\nu}$ are the zeros of $\beta(x;\nu)$ such that $\xi_{1,\nu} \neq \xi_{2,\nu}$, $\xi_{1,\nu} \neq \xi_{2,\nu} \star \nu$, $\xi_{2,\nu} \neq \xi_{1,\nu} \star \nu$, equivalently, $\xi_{1,\nu}$ and $\xi_{2,\nu}$ are the zeros of $\beta(x;\nu)$ such that $\xi_{1,\nu} \neq \xi_{2,\nu}$, $(D_\nu \beta)(\xi_{1,\nu} \star \nu^*) \neq 0$ and $(D_\nu \beta)(\xi_{2,\nu} \star \nu^*) \neq 0$. In this case, (see Theorem 3.1.10) we will prove that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ν-coherent pair of regular linear functionals, then $\mathcal{V}$ and $\mathcal{U}$ are D_ν-semiclassical of class at most 1 and 5 respectively.

iii. ξ_ν is a double zero of $\beta(x;\nu)$. In this case two situations must be analyzed: either $(D_\nu \beta)(\xi_\nu \star \nu^*;\nu) = 0$ or $(D_\nu \beta)(\xi_\nu \star \nu^*;\nu) \neq 0$. If $(D_\nu \beta)(\xi_\nu \star \nu^*;\nu) = 0$ and $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ν-coherent pair of regular linear functionals, then we will prove that $\mathcal{U}$ and $\mathcal{V}$ are D_ν-semiclassical of class at most 1 and 5 respectively (see Remark 3.1.11). If $(D_\nu \beta)(\xi_\nu \star \nu^*;\nu) \neq 0$ and $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ν-coherent pair of regular linear functionals, then we will show that $\mathcal{V}$ and $\mathcal{U}$ are D_ν-semiclassical of class at most 1 and 5 respectively (see Theorem 3.1.12).

Furthermore, in all cases, we will also state that $\mathcal{U}$ and $\mathcal{V}$ are related by the following expression of rational type

$$\tilde{\sigma}(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V}, \quad \deg(\tilde{\sigma}(x;\nu)) \leq 3, \ \deg(\rho(x;\nu)) = 1.$$

Theorem 3.1.7. *Let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1 \neq b_1$. If ξ_ν and $\xi_\nu \star \nu^*$ are the zeros of $\beta(x;\nu)$, then there exist polynomials $\tilde{\alpha}_3(x;\nu)$, $\varphi_2(x;\nu)$, and $\gamma_1(x;\nu)$ of degrees $\leq 3, \leq 2$, and 1, respectively, such that*

$$D_{\nu^*}\left[\tilde{\alpha}_3(x;\nu)\mathcal{U}\right] = -\varphi_2(x;\nu)\mathcal{U}, \tag{3.1.17}$$

$$\tilde{\alpha}_3(x;\nu)\mathcal{U} = \gamma_1(x;\nu)\mathcal{V}. \tag{3.1.18}$$

Thus, $\mathcal{U}$ is D_ν-semiclassical of class at most 1 and $\mathcal{V}$ is D_ν-semiclassical of class at most 5.

Proof. From (3.1.16),

$$\gamma_1(x;\nu) = x - (\xi_\nu \star \nu^*) \quad \text{and} \quad \beta(x;\nu) = \hbar_{\nu^*}(x - \xi_\nu)\gamma_1(x;\nu).$$

Then, from (3.1.11) we get $\gamma_2(\xi_\nu \star \nu^*;\nu) = 0$ and thus

$$\gamma_2(x;\nu) = \gamma_1(x;\nu)\varrho_1(x;\nu),$$

where $\varrho_1(x;\nu)$ is a monic polynomial of degree 1. As a consequence,

$$(D_{\nu^*}\gamma_2)(x;\nu) = \gamma_1(x;\nu) + \varrho_1(x \star \nu^*;\nu),$$

and, from (3.1.10) we obtain $\alpha(\xi_\nu;\nu)=0$ and

$$\alpha(x;\nu)=\gamma_1(x\star\nu^*;\nu)\tilde{\alpha}_3(x;\nu)=\hbar_{\nu^*}(x-\xi_\nu)\tilde{\alpha}_3(x;\nu),$$

where

$$\tilde{\alpha}_3(x;\nu)=\hbar_\nu[\varrho_1(x\star\nu^*;\nu)\varphi_2(x;\nu)-\varphi_3(x;\nu)].$$

Thus, (3.1.14) and (3.1.15) become

$$\begin{aligned}\gamma_2(x\star\nu^*;\nu)D_{\nu^*}\mathcal{V}+\hbar_{\nu^*}\varrho_1(x\star\nu^*;\nu)\mathcal{V}&=-\varrho_1(x\star\nu^*;\nu)\varphi_2(x;\nu)\mathcal{U},\\ \gamma_2(x\star\nu^*;\nu)D_{\nu^*}\mathcal{V}+\hbar_{\nu^*}\left[\gamma_1(x;\nu)+\varrho_1(x\star\nu^*;\nu)\right]\mathcal{V}&=-\varphi_3(x;\nu)\mathcal{U}.\end{aligned}$$

By elimination of $\gamma_2(x\star\nu^*;\nu)D_{\nu^*}\mathcal{V}$ we get (3.1.18), and, if we take D_{ν^*} in (3.1.18), then (3.1.17) follows from (3.1.5). Therefore, $\mathcal{U}$ is D_{ν^*}-semiclassical of class at most 1 and, as a consequence, using Proposition 1.4.7 and Proposition 1.4.8, we get the desired result. $\square$

For the second case we need some previous results which will be stated as lemmas.

Lemma 3.1.8. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1\neq b_1$, let $\alpha(x;\nu)$, $\beta(x;\nu)$, and $\phi(x;\nu)$ be the polynomials given in Corollary 3.1.5, and let ξ_ν be a zero of $\beta(x;\nu)$.*

- ***i.*** *If $\beta(\xi_\nu\star\nu^*;\nu)\neq 0$ and $\alpha(\xi_\nu;\nu)=0$, then $\phi(\xi_\nu;\nu)=0$ and $\gamma_1(\xi_\nu\star\nu^*;\nu)\neq 0$.*
- ***ii.*** *If $\alpha(\xi_\nu;\nu)\neq 0$, then there exists a nonzero constant C_ν, independent on n, such that*

$$\gamma_n(\xi_\nu\star\nu^*;\nu)+C_\nu\left(D_{\nu^*}\gamma_n\right)(\xi_\nu;\nu)=0,\quad n\geq 1, \tag{3.1.19}$$

where $\gamma_n(x;\nu)$ is the monic polynomial of degree n introduced in Lemma 3.1.4.

Proof. ***i.*** From (3.1.16) it follows that $\gamma_1(\xi_\nu\star\nu^*;\nu)\neq 0$ and, from (3.1.13) for $n=1$, we get $\phi(\xi_\nu;\nu)=0$.

ii. $\phi(\xi_\nu;\nu)\neq 0$ follows from (3.1.13) for $n=1$, and $\alpha(\xi_\nu;\nu)\neq 0$. As a consequence, if $C_\nu=\hbar_{\nu^*}\alpha(\xi_\nu;\nu)/\phi(\xi_\nu;\nu)$, then we get desired result from (3.1.13). $\square$

Lemma 3.1.9. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1\neq b_1$, and let $\gamma_n(x;\nu)$ be the monic polynomial of degree n introduced in Lemma 3.1.4. If there exist constants $\xi_{1,\nu},\xi_{2,\nu},C_{1,\nu},C_{2,\nu}$ independent on n, such that $\xi_{1,\nu}\neq\xi_{2,\nu}\star\nu$, $\xi_{2,\nu}\neq\xi_{1,\nu}\star\nu$, and*

$$\gamma_n(\xi_{k,\nu}\star\nu^*;\nu)+C_{k,\nu}\left(D_{\nu^*}\gamma_n\right)(\xi_{k,\nu};\nu)=0,\quad k=1,2,\quad n\geq 1, \tag{3.1.20}$$

then $\xi_{1,\nu}=\xi_{2,\nu}$ and $C_{1,\nu}=C_{2,\nu}$.

Proof. From (3.1.4) and (3.1.20),

$$Q_n(\xi_{1,\nu} \star \nu^*) + C_{1,\nu} \left(D_{\nu^*} Q_n\right)(\xi_{1,\nu}) = Q_n(\xi_{2,\nu} \star \nu^*) + C_{2,\nu} \left(D_{\nu^*} Q_n\right)(\xi_{2,\nu}), \quad n \geq 0.$$

Thus, since $\{Q_n(x)\}_{n\geq 0}$ is a basis of $\mathbb{P}$, we obtain

$$p(\xi_{1,\nu} \star \nu^*) + C_{1,\nu} \left(D_{\nu^*} p\right)(\xi_{1,\nu}) = p(\xi_{2,\nu} \star \nu^*) + C_{2,\nu} \left(D_{\nu^*} p\right)(\xi_{2,\nu}), \quad \forall p \in \mathbb{P}. \tag{3.1.21}$$

If $\xi_{1,\nu} = \xi_{2,\nu}$, we can choose $p(x) = x$, and hence $C_{1,\nu} = C_{2,\nu}$ holds. On the other hand, let

$$p(x;\nu) = (x - \xi_{2,\nu})^n (x - \xi_{2,\nu} \star \nu^*)^n, \quad n \geq 1.$$

Then

$$\begin{aligned}\left(D_{\nu^*} p\right)(x;\nu) &= (x - \xi_{2,\nu})^n D_{\nu^*}\left[(x - \xi_{2,\nu} \star \nu^*)^n\right] + (x \star \nu^* - \xi_{2,\nu} \star \nu^*)^n D_{\nu^*}\left[(x - \xi_{2,\nu})^n\right] \\ &= (x - \xi_{2,\nu})^n \left\{ D_{\nu^*}\left[(x - \xi_{2,\nu} \star \nu^*)^n\right] + \hbar_{\nu^*}^n D_{\nu^*}\left[(x - \xi_{2,\nu})^n\right] \right\},\end{aligned}$$

and, as a consequence, (3.1.21) becomes

$$\begin{aligned}(\xi_{1,\nu} - \xi_{2,\nu})^n \Bigg\{ &\hbar_{\nu^*}^n (\xi_{1,\nu} \star \nu^* - \xi_{2,\nu})^n \\ &+ C_{1,\nu}\, D_{\nu^*}\Big[(x - \xi_{2,\nu} \star \nu^*)^n + \hbar_{\nu^*}^n (x - \xi_{2,\nu})^n\Big]\Big|_{x=\xi_{1,\nu}} \Bigg\} = 0, \quad n \geq 1.\end{aligned} \tag{3.1.22}$$

If $\xi_{1,\nu} \neq \xi_{2,\nu}$, then from (3.1.22) for $n = 1$, it follows that

$$C_{1,\nu} = -\frac{\hbar_{\nu^*}(\xi_{1,\nu} \star \nu^* - \xi_{2,\nu})}{1 + \hbar_{\nu^*}}.$$

Replacing this value in (3.1.22) for $n = 2$, we get

$$\begin{aligned}0 &= \hbar_{\nu^*}^2 (\xi_{1,\nu} \star \nu^* - \xi_{2,\nu})^2 - \frac{\hbar_{\nu^*}(\xi_{1,\nu} \star \nu^* - \xi_{2,\nu})}{1 + \hbar_{\nu^*}} \\ &\qquad \Big\{ (\xi_{1,\nu} - \xi_{2,\nu} \star \nu^*) + (\xi_{1,\nu} \star \nu^* - \xi_{2,\nu} \star \nu^*) + \hbar_{\nu^*}^2 \big[(\xi_{1,\nu} - \xi_{2,\nu}) + (\xi_{1,\nu} \star \nu^* - \xi_{2,\nu}) \big] \Big\} \\ &= \frac{\hbar_{\nu^*}(\xi_{1,\nu} \star \nu^* - \xi_{2,\nu})}{1 + \hbar_{\nu^*}} \Big[\hbar_{\nu^*}(\xi_{1,\nu} \star \nu^* - \xi_{2,\nu}) - (\xi_{1,\nu} - \xi_{2,\nu} \star \nu^*) \\ &\qquad - (1 + \hbar_{\nu^*})\hbar_{\nu^*}(\xi_{1,\nu} - \xi_{2,\nu}) \Big] = \hbar_{\nu^*}(\xi_{1,\nu} \star \nu^* - \xi_{2,\nu})(\xi_{2,\nu} \star \nu^* - \xi_{1,\nu}),\end{aligned}$$

which contradicts the hypothesis $\xi_{1,\nu} \neq \xi_{2,\nu} \star \nu$, $\xi_{2,\nu} \neq \xi_{1,\nu} \star \nu$. Therefore $\xi_{1,\nu} = \xi_{2,\nu}$. □

Theorem 3.1.10. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1 \neq b_1$, and let $\beta(x;\nu)$ be the polynomial given by (3.1.11). If $\xi_{1,\nu}$ and $\xi_{2,\nu}$ are the zeros of $\beta(x;\nu)$ such that $\xi_{1,\nu}\neq\xi_{2,\nu}, \xi_{1,\nu}\neq\xi_{2,\nu}\star\nu, \xi_{2,\nu}\neq\xi_{1,\nu}\star\nu$, then*

$$\tilde{\alpha}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)\mathcal{V}, \tag{3.1.23}$$

$$\tilde{\alpha}(x;\nu)D_{\nu^*}\mathcal{V} = \tilde{\phi}(x;\nu)\mathcal{V}, \tag{3.1.24}$$

$$\tilde{\phi}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)D_{\nu^*}\mathcal{V}, \tag{3.1.25}$$

where $\tilde{\beta}(x;\nu) = \hbar_{\nu^*}(x-\xi)$ for some $\xi\in\{\xi_{1,\nu},\xi_{2,\nu}\}$, $\deg(\tilde{\alpha}(x;\nu))\leq 3$, and $\deg(\tilde{\phi}(x;\nu))\leq 2$. Moreover,

$$D_{\nu^*}\left[\tilde{\alpha}(x;\nu)\mathcal{V}\right] = \left(\tilde{\phi}(x\star\nu^*;\nu) + \hbar_{\nu^*}\left(D_{\nu^*}\tilde{\alpha}\right)(x;\nu)\right)\mathcal{V}. \tag{3.1.26}$$

Therefore, $\mathcal{V}$ and $\mathcal{U}$ are D_ν-semiclassical of class at most 1 and 5, respectively.

Proof. Let $\alpha(x;\nu), \beta(x;\nu)$, and $\phi(x;\nu)$ be the polynomials given in Corollary 3.1.5 and let

$$\beta(x;\nu) = \hbar_{\nu^*}(x-\xi_{1,\nu})(x-\xi_{2,\nu}) = (x-\xi_{1,\nu})\tilde{\beta}(x;\nu).$$

Since $\xi_{1,\nu}\neq\xi_{2,\nu}, \xi_{1,\nu}\neq\xi_{2,\nu}\star\nu, \xi_{2,\nu}\neq\xi_{1,\nu}\star\nu$, from Lemmas 3.1.9 and 3.1.8.*ii.* it follows that either $\alpha(\xi_{1,\nu};\nu)=0$ or $\alpha(\xi_{2,\nu};\nu)=0$. Assuming $\alpha(\xi_{1,\nu};\nu)=0$, then

$$\alpha(x;\nu) = (x-\xi_{1,\nu})\tilde{\alpha}(x;\nu),$$

and from Lemma 3.1.8.*i.* we get $\phi(\xi_{1,\nu};\nu)=0$ and $\gamma_1(\xi_{1,\nu}\star\nu^*;\nu)\neq 0$, so

$$\phi(x;\nu) = (x-\xi_{1,\nu})\tilde{\phi}(x;\nu).$$

Therefore, (3.1.7)-(3.1.9) and (3.1.13) become

$$\tilde{\alpha}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)\mathcal{V} + M_1\delta_{\xi_{1,\nu}}, \tag{3.1.27}$$

$$\tilde{\alpha}(x;\nu)D_{\nu^*}\mathcal{V} = \tilde{\phi}(x;\nu)\mathcal{V} + M_2\delta_{\xi_{1,\nu}}, \tag{3.1.28}$$

$$\tilde{\phi}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)D_{\nu^*}\mathcal{V} + M_3\delta_{\xi_{1,\nu}}, \tag{3.1.29}$$

$$\tilde{\phi}(x;\nu)\gamma_n(x\star\nu^*;\nu) + \hbar_{\nu^*}\tilde{\alpha}(x;\nu)\left(D_{\nu^*}\gamma_n\right)(x;\nu) = -\varphi_{n+1}(x;\nu)\tilde{\beta}(x;\nu), \quad n\geq 1. \tag{3.1.30}$$

Hence, for $n\geq 1$,

$$\begin{aligned}\Big(\tilde{\phi}(x;\nu)\gamma_n(x\star\nu^*;\nu) + \hbar_{\nu^*}\tilde{\alpha}(x;\nu)\left(D_{\nu^*}\gamma_n\right)(x;\nu)\Big)\mathcal{U} &\overset{(3.1.30)}{\underset{(3.1.5)}{=}} \tilde{\beta}(x;\nu)D_{\nu^*}[\gamma_n(x;\nu)\mathcal{V}] \\ &\overset{(3.1.27)}{\underset{(3.1.29)}{=}} \gamma_n(x\star\nu^*;\nu)\left(\tilde{\phi}(x;\nu)\mathcal{U} - M_3\delta_{\xi_{1,\nu}}\right) + \hbar_{\nu^*}\left(D_{\nu^*}\gamma_n\right)(x;\nu)\Big(\tilde{\alpha}(x;\nu)\mathcal{U} - M_1\delta_{\xi_{1,\nu}}\Big),\end{aligned}$$

then,

$$M_3\gamma_n(\xi_{1,\nu}\star\nu^*;\nu) = -\hbar_{\nu^*}M_1\left(D_{\nu^*}\gamma_n\right)(\xi_{1,\nu};\nu), \quad n\geq 1. \tag{3.1.31}$$

Since $(D_{\nu^*}\gamma_1)(\xi_{1,\nu};\nu) = 1$ and $\gamma_1(\xi_{1,\nu}\star\nu^*;\nu)\neq 0$, $M_1 = 0$ if and only if $M_3 = 0$. If $M_3 = 0$, then (3.1.23) and (3.1.25) hold. If $M_3 \neq 0$ and $\tilde{\alpha}(\xi_{2,\nu};\nu)\neq 0$, then $\alpha(\xi_{2,\nu};\nu)\neq 0$ and, from Lemma 3.1.8.*ii*., there exists $C_\nu \neq 0$, independent on n, such that

$$\gamma_n(\xi_{2,\nu}\star\nu^*;\nu) + C_\nu\,(D_{\nu^*}\gamma_n)(\xi_{2,\nu};\nu) = 0, \quad n\geq 1.$$

But, since $\xi_{1,\nu}\neq\xi_{2,\nu}, \xi_{1,\nu}\neq\xi_{2,\nu}\star\nu, \xi_{2,\nu}\neq\xi_{1,\nu}\star\nu$, then from Lemma 3.1.9 the previous identity or (3.1.31) does not hold, which is a contradiction. On the other hand, if $M_3\neq 0$ and $\tilde{\alpha}(\xi_{2,\nu};\nu) = 0$, we can do the same analysis as for $\xi_{1,\nu}$ and we obtain

$$\tilde{M}_3\gamma_n(\xi_{2,\nu}\star\nu^*;\nu) = -\hbar_{\nu^*}\tilde{M}_1\,(D_{\nu^*}\gamma_n)(\xi_{2,\nu};\nu), \quad n\geq 1. \tag{3.1.32}$$

Thus, $\tilde{M}_1 = 0$ if and only if $\tilde{M}_3 = 0$. If $\tilde{M}_3 = 0$, then we get (3.1.23) and (3.1.25). If $\tilde{M}_3 \neq 0$, then from Lemma 3.1.9 neither (3.1.31) nor (3.1.32) hold, which is a contradiction. So $\tilde{M}_3 = 0$.

We suppose that $M_3 = 0$ (if $\tilde{M}_3 = 0$, the following computations are true for $\xi_{2,\nu}$). Then,

$$\begin{aligned}
-\varphi_2(x;\nu)\tilde{\beta}(x;\nu)\mathcal{V} &\underset{(3.1.5)}{\overset{(3.1.23)}{=}} \tilde{\alpha}(x;\nu)D_{\nu^*}[\gamma_1(x;\nu)\mathcal{V}] \\
&\underset{(3.1.30)}{\overset{(3.1.28)}{=}} \gamma_1(x\star\nu^*;\nu)M_2\delta_{\xi_{1,\nu}} - \varphi_2(x;\nu)\tilde{\beta}(x;\nu)\mathcal{V}.
\end{aligned}$$

But $\gamma_1(\xi_{1,\nu}\star\nu^*;\nu)\neq 0$, thus $M_2 = 0$ and, as a consequence, (3.1.24) holds. Finally, using (3.1.24) we get (3.1.26), and, as a consequence, $\mathcal{V}$ is D_{ν^*}-semiclassical of class at most 1. Therefore, from Proposition 1.4.7 and Proposition 1.4.8, the proof is complete. $\square$

Remark 3.1.11. If $\beta(x;\nu)$ has a double zero ξ_ν, then there are two possibilities: either $(D_\nu\beta)(\xi_\nu\star\nu^*;\nu) = 0$ or $(D_\nu\beta)(\xi_\nu\star\nu^*;\nu)\neq 0$. If $(D_\nu\beta)(\xi_\nu\star\nu^*;\nu) = 0$, from (3.1.16) it follows that

- when $\nu = \omega$, $\omega = 0$ which is a contradiction, and thus this case it is not possible;
- when $\nu = q$, $\xi_q = 0$, and therefore in this case Theorem 3.1.7 holds.

When $(D_\nu\beta)(\xi_\nu\star\nu^*;\nu)\neq 0$, we have the following theorem.

Theorem 3.1.12. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ν-coherent pair given by (3.1.1) such that $a_1\neq b_1$, and let $\beta(x;\nu)$ be the polynomial given by (3.1.11). If ξ_ν is a double zero of $\beta(x;\nu)$ such that $(D_\nu\beta)(\xi_\nu\star\nu^*;\nu)\neq 0$, and there exists $N\geq 2$ such that*

$$\gamma_N(\xi_\nu\star\nu^*;\nu) - \gamma_1(\xi_\nu\star\nu^*;\nu)\,(D_{\nu^*}\gamma_N)(\xi_\nu;\nu)\neq 0,$$

with $\gamma_n(x;\nu)$ the monic polynomial of degree n given in Lemma 3.1.4, then

$$\tilde{\alpha}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)\mathcal{V}, \tag{3.1.33}$$

$$\tilde{\alpha}(x;\nu)D_{\nu^*}\mathcal{V} = \tilde{\phi}(x;\nu)\mathcal{V}, \tag{3.1.34}$$
$$\tilde{\phi}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)D_{\nu^*}\mathcal{V}, \tag{3.1.35}$$

where $\tilde{\beta}(x;\nu) = \hbar_{\nu^*}(x-\xi_\nu)$, $\deg(\tilde{\alpha}(x;\nu)) \leq 3$, *and* $\deg(\tilde{\phi}(x;\nu)) \leq 2$. *Besides,*

$$D_{\nu^*}\left[\tilde{\alpha}(x;\nu)\mathcal{V}\right] = \Big(\tilde{\phi}(x\star\nu^*;\nu) + \hbar_{\nu^*}\left(D_{\nu^*}\tilde{\alpha}\right)(x;\nu)\Big)\mathcal{V}. \tag{3.1.36}$$

Therefore, $\mathcal{V}$ *and* $\mathcal{U}$ *are* D_ν*-semiclassical of class at most* 1 *and* 5*, respectively.*

Proof. Let $\alpha(x;\nu), \beta(x;\nu)$, and $\phi(x;\nu)$ be the polynomials given in Corollary 3.1.5. From (3.1.16) we get $\gamma_1(\xi_\nu\star\nu^*;\nu) \neq 0$. Since

$$\gamma_1(\xi_\nu\star\nu^*;\nu) + C_\nu\left(D_{\nu^*}\gamma_1\right)(\xi_\nu;\nu) \neq 0, \quad \text{for all constant } C_\nu \neq -\gamma_1(\xi_\nu\star\nu^*;\nu),$$

and for $C_\nu = -\gamma_1(\xi_\nu\star\nu^*;\nu)$, by hypothesis there exists $N \geq 2$ such that

$$\gamma_N(\xi_\nu\star\nu^*;\nu) + C_\nu\left(D_{\nu^*}\gamma_N\right)(\xi_\nu;\nu) \neq 0, \quad C_\nu = -\gamma_1(\xi_\nu\star\nu^*;\nu) \text{ and for some } N \geq 2,$$

then, from Lemma 3.1.8.*ii.* we obtain $\alpha(\xi_\nu;\nu) = 0$, and from Lemma 3.1.8.*i.* it follows that $\phi(\xi_\nu;\nu) = 0$. Thus,

$$\beta(x;\nu) = (x-\xi_\nu)\tilde{\beta}(x;\nu), \quad \alpha(x;\nu) = (x-\xi_\nu)\tilde{\alpha}(x;\nu), \quad \phi(x;\nu) = (x-\xi_\nu)\tilde{\phi}(x;\nu).$$

Thus (3.1.7) - (3.1.9) and (3.1.13) become

$$\tilde{\alpha}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)\mathcal{V} + \widetilde{M}_1\delta_{\xi_\nu}, \tag{3.1.37}$$
$$\tilde{\alpha}(x;\nu)D_{\nu^*}\mathcal{V} = \tilde{\phi}(x;\nu)\mathcal{V} + \widetilde{M}_2\delta_{\xi_\nu}, \tag{3.1.38}$$
$$\tilde{\phi}(x;\nu)\mathcal{U} = \tilde{\beta}(x;\nu)D_{\nu^*}\mathcal{V} + \widetilde{M}_3\delta_{\xi_\nu}, \tag{3.1.39}$$
$$\tilde{\phi}(x;\nu)\gamma_n(x\star\nu^*;\nu) + \hbar_{\nu^*}\tilde{\alpha}(x;\nu)\left(D_{\nu^*}\gamma_n\right)(x;\nu) = -\varphi_{n+1}(x;\nu)\tilde{\beta}(x;\nu), \quad n \geq 1. \tag{3.1.40}$$

Consequently, for $n \geq 1$,

$$\begin{aligned}\Big(\tilde{\phi}(x;\nu)\gamma_n(x\star\nu^*;\nu) + \hbar_{\nu^*}\tilde{\alpha}(x;\nu)\left(D_{\nu^*}\gamma_n\right)(x;\nu)\Big)\mathcal{U} &\underset{(3.1.5)}{\overset{(3.1.40)}{=}} \tilde{\beta}(x;\nu)D_{\nu^*}\left[\gamma_n(x;\nu)\mathcal{V}\right] \\ &\underset{(3.1.39)}{\overset{(3.1.37)}{=}} \gamma_n(x\star\nu^*;\nu)\left(\tilde{\phi}(x;\nu)\mathcal{U} - \widetilde{M}_3\delta_{\xi_\nu}\right) + \hbar_{\nu^*}\left(D_{\nu^*}\gamma_n\right)(x;\nu)\Big(\tilde{\alpha}(x;\nu)\mathcal{U} - \widetilde{M}_1\delta_{\xi_\nu}\Big),\end{aligned}$$

and, therefore,

$$\widetilde{M}_3\gamma_n(\xi_\nu\star\nu^*;\nu) + \hbar_{\nu^*}\widetilde{M}_1\left(D_{\nu^*}\gamma_n\right)(\xi_\nu;\nu) = 0, \quad n \geq 1. \tag{3.1.41}$$

But $\left(D_{\nu^*}\gamma_1\right)(\xi_\nu;\nu) = 1$ and $\gamma_1(\xi_\nu\star\nu^*;\nu) \neq 0$, hence $\widetilde{M}_1 = 0$ if and only if $\widetilde{M}_3 = 0$. If $\widetilde{M}_3 \neq 0$, from (3.1.41) for $n = 1$, it follows that $\hbar_{\nu^*}\widetilde{M}_1/\widetilde{M}_3 = -\gamma_1(\xi_\nu\star\nu^*;\nu)$. Then

$$\gamma_n(\xi_\nu\star\nu^*;\nu) - \gamma_1(\xi_\nu\star\nu^*;\nu)\left(D_{\nu^*}\gamma_n\right)(\xi_\nu;\nu) = 0, \quad n \geq 1,$$

which yields a contradiction. So $\widetilde{M}_3 = 0$ and, as a consequence, (3.1.33) and (3.1.35) follows. Besides,

$$\begin{aligned}-\varphi_2(x;\nu)\tilde{\beta}(x;\nu)\mathcal{V} &\underset{(3.1.5)}{\overset{(3.1.33)}{=}} \tilde{\alpha}(x;\nu)D_{\nu^*}[\gamma_1(x;\nu)\mathcal{V}] \\ &\underset{(3.1.40)}{\overset{(3.1.38)}{=}} \gamma_1(x\star\nu^*;\nu)\widetilde{M}_2\delta_{\xi_\nu} - \varphi_2(x;\nu)\tilde{\beta}(x;\nu)\mathcal{V}.\end{aligned}$$

Since $\gamma_1(\xi_\nu\star\nu^*;\nu)\neq 0$, it follows that $\widetilde{M}_2 = 0$ and then (3.1.34) holds. Finally, from (3.1.34), (3.1.36) follows, and hence $\mathcal{V}$ is D_{ν^*}-semiclassical of class at most 1. As a consequence, from Proposition 1.4.7 and Proposition 1.4.8, we deduce the desired result. □

Remark 3.1.13. When $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-D_ν-coherent pair of regular linear functionals such that $a_1\neq b_1$, all the previous results of this section hold with the following modifications

- (3.1.3) holds for $m\geq n\geq 1$, and $\gamma_n(x;\nu)$ will be denoted by $\gamma_n^{(1,0)}(x;\nu)$. Besides, if we choose $A_{1,n,\nu}=\cdots=A_{n-1,n,\nu}=0$ in the proof of Lemma 3.1.2, then (3.1.4) is replaced by
$$\gamma_n^{(1,0)}(x;\nu) = Q_n(x) + \frac{(-1)^{n+1}\langle\mathcal{V},Q_n^2(x)\rangle}{a_n\cdots a_3a_2a_1\langle\mathcal{V},1\rangle},\quad n\geq 1. \tag{3.1.42}$$
- In Lemma 3.1.4, (3.1.6) is replaced by
$$\varphi_{n+1}^{(1,0)}(x;\nu) = \sum_{k=0}^{n-1}\frac{\eta_{k,1,\nu}\langle\gamma_n^{(1,0)}(x;\nu)\mathcal{V},P_k^{[1,\nu]}(x)\rangle}{\langle\mathcal{U},P_{k+1}^2(x)\rangle}P_{k+1}(x),\quad n\geq 1.$$
- Note that $\deg(\gamma_n^{(1,0)}(x;\nu)) = n$, and $\deg(\varphi_{n+1}^{(1,0)}(x;\nu))\leq n$.
- The degrees of the polynomials $\alpha^{(1,0)}(x;\nu)$, $\phi^{(1,0)}(x;\nu)$, and $\beta^{(1,0)}(x;\nu)$ introduced in Corollary 3.1.5 are ≤ 3, ≤ 2, and 2, respectively.
- (Remark 3.1.6) From (3.1.42), the leading coefficients of $\varphi_2^{(1,0)}(x;\nu)$ and $\varphi_3^{(1,0)}(x;\nu)$ are, respectively
$$\frac{\left\langle\mathcal{V},\gamma_1^{(1,0)}(x;\nu)\right\rangle}{\langle\mathcal{U},P_1^2(x)\rangle}\overset{(3.1.42)}{=}\frac{\langle\mathcal{V},Q_1^2(x)\rangle}{a_1\langle\mathcal{U},P_1^2(x)\rangle},$$
$$\eta_{1,1,\nu}\frac{\langle\mathcal{V},\gamma_2^{(1,0)}(x;\nu)P_1^{[1,\nu]}(x)\rangle}{\langle\mathcal{U},P_2^2(x)\rangle}\underset{(3.1.42)}{\overset{(3.1.2)}{=}}\eta_{1,1,\nu}\frac{\left\langle\mathcal{V},Q_2^2(x)\right\rangle}{a_2\langle\mathcal{U}.P_2^2(x)\rangle}.$$
Hence, $\deg\left(\varphi_2^{(1,0)}(x;\nu)\right) = 1$ and $\deg\left(\varphi_3^{(1,0)}(x;\nu)\right) = 2$. Besides,
$$\operatorname{leadcoeff}(\beta^{(1,0)}(x;\nu)) = \operatorname{leadcoeff}(\beta(x;\nu)),$$

and the leading coefficients of polynomials $\alpha^{(1,0)}(x;\nu)$ and $\phi^{(1,0)}(x;\nu)$ (see (3.1.10)-(3.1.12)) are, respectively

$$\hbar_{\nu^*}\frac{\langle\mathcal{V},Q_1^2(x)\rangle}{a_1\langle\mathcal{U},P_1^2(x)\rangle}-\eta_{1,1,\nu}\frac{\langle\mathcal{V},Q_2^2(x)\rangle}{a_2\langle\mathcal{U},P_2^2(x)\rangle},$$
$$\eta_{1,1,\nu}\frac{\langle\mathcal{V},Q_2^2(x)\rangle}{a_2\langle\mathcal{U},P_2^2(x)\rangle}-\eta_{1,1,\nu}\frac{\langle\mathcal{V},Q_1^2(x)\rangle}{a_1\langle\mathcal{U},P_1^2(x)\rangle}.$$

- In Theorem 3.1.7, $\deg(\tilde{\alpha}_3^{(1,0)}(x;\nu))\leq 2$, $\deg\left(\varphi_2^{(1,0)}(x;\nu)\right)=1$, and $\deg(\gamma_1^{(1,0)}(x;\nu))=1$. Thus, $\mathcal{U}$ is a D_{ν^*}-classical linear functional and as a consequence (see Theorem 1.5.3), $\tilde{\alpha}_3^{(1,0)}(x;\nu)\mathcal{U}$ is also D_ν^*-classical. Therefore, from Propositions 1.4.7 and 1.4.8, $\mathcal{U}$ is a D_ν-classical linear functional and $\mathcal{V}$ is a D_ν-semiclassical linear functional of class at most 1.

- In Theorems 3.1.10 and 3.1.12, $\deg(\tilde{\alpha}^{(1,0)}(x;\nu))\leq 2$ and $\deg(\tilde{\phi}^{(1,0)}(x;\nu))\leq 1$. Thus, from Propositions 1.4.7 and 1.4.8, $\mathcal{V}$ is a D_ν-classical linear functional and $\mathcal{U}$ is a D_ν-semiclassical linear functional of class at most 3.

- Notice that in these three Theorems we also get

$$\tilde{\sigma}(x;\nu)\mathcal{U}=\rho(x;\nu)\mathcal{V},\quad \deg\left(\tilde{\sigma}(x;\nu)\right)\leq 2,\, \deg\left(\rho(x;\nu)\right)=1.$$

3.1.1 The Case When $\mathcal{U}$ is D_ν-Classical

In this subsection, we will study the case when $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-D_ν-coherent or $(1,1)$-D_ν-coherent pair of regular linear functionals and $\mathcal{U}$ is a D_ν-classical (equivalently, D_{ν^*}-classical) linear functional given by

$$D_{\nu^*}\left[\sigma(x;\nu)\mathcal{U}\right]=\tau(x;\nu)\mathcal{U},\quad \deg(\sigma(x;\nu))\leq 2,\, \deg(\tau(x;\nu))=1. \tag{3.1.43}$$

Proposition 3.1.14. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,0)$-D_ν-coherent pair given by*

$$P_n^{[1,\nu]}(x)+a_nP_{n-1}^{[1,\nu]}(x)=Q_n(x),\quad a_n\neq 0,\, n\geq 1. \tag{3.1.44}$$

If $\mathcal{U}$ is a D_ν-classical linear functional given by (3.1.43), then

$$\sigma(x;\nu)\mathcal{U}=\frac{a_1\langle\mathcal{U},\sigma(x;\nu)\rangle}{\langle\mathcal{V},Q_1^2(x)\rangle}\left[Q_1(x)+\frac{\langle\mathcal{V},Q_1^2(x)\rangle}{a_1\langle\mathcal{V},1\rangle}\right]\mathcal{V}, \tag{3.1.45}$$

and, therefore, $\mathcal{V}$ is a D_ν-semiclassical linear functional of class at most 1.

Proof. From Theorem 1.5.3, we have that $\{P_n^{[1,\nu]}(x)\}_{n\geq 0}$ is also a D_ν-classical SMOP with respect to $\mathcal{U}^{[1,\nu]}=\sigma(x;\nu)\mathcal{U}$.

Let $\{\mathfrak{q}_n\}_{n\geq 0}$ and $\{\mathfrak{e}_{n,1,\nu}\}_{n\geq 0}$ be the corresponding dual bases of the SMOP $\{Q_n(x)\}_{n\geq 0}$ and $\{P_n^{[1,\nu]}(x)\}_{n\geq 0}$. Then

$$\mathfrak{e}_{n,1,\nu}=\sum_{k\geq 0}\langle \mathfrak{e}_{n,1,\nu}, Q_k(x)\rangle\, \mathfrak{q}_k \overset{(3.1.44)}{=} \mathfrak{q}_n + a_{n+1}\mathfrak{q}_{n+1}, \quad n\geq 0. \tag{3.1.46}$$

Thus,

$$\frac{\mathcal{U}^{[1,\nu]}}{\langle \mathcal{U}^{[1,\nu]},1\rangle} \overset{(1.3.5)}{=} \mathfrak{e}_{0,1,\nu} \underset{n=0}{\overset{(3.1.46)}{=}} \mathfrak{q}_0 + a_1\mathfrak{q}_1 \overset{(1.3.5)}{=} \frac{\mathcal{V}}{\langle \mathcal{V},1\rangle} + a_1\frac{Q_1(x)\mathcal{V}}{\langle \mathcal{V}, Q_1^2(x)\rangle} \underset{n=1}{\overset{(3.1.42)}{=}} \frac{a_1\gamma_1^{(1,0)}(x)\mathcal{V}}{\langle \mathcal{V}, Q_1^2(x)\rangle},$$

which is (3.1.45), and from Proposition 1.4.8 the statement follows. □

Theorem 3.1.15. *([4, p. 314]) Let $\{T_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be two SMOP with respect to $\mathcal{W}$ and $\mathcal{V}$, respectively. Then, the following statements are equivalent*

- ***i)*** *There exist complex sequences $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$, with $a_1\neq b_1$, $a_nb_n\neq 0$, $n\geq 1$, and such that*
$$T_n(x)+a_nT_{n-1}(x)=Q_n(x)+b_nQ_{n-1}(x), \quad n\geq 1.$$
- ***ii)*** *$T_n(x)\neq Q_n(x)$, for $n\geq 1$, and there exist constants C^T, C^Q, and κ such that*
$$(x-C^T)\mathcal{W}=\kappa(x-C^Q)\mathcal{V}.$$

Remark 3.1.16. Let $\{P_n(x)\}_{n\geq 0}$ be a D_ν-classical SMOP with respect to a linear functional $\mathcal{U}$ which satisfies (3.1.43). Let $\{P_n^{[1,\nu]}(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be the SPOM with respect to $\mathcal{U}^{[1,\nu]}=\sigma(x;\nu)\mathcal{U}$ and $\mathcal{V}$, respectively, and corresponding TTRR given as in (1.3.1). Then, from Theorem 3.1.15 (see [4]) we have the following results:

- In the proof of $(i)\Rightarrow(ii)$ we get that the condition $b_n\neq 0$, $n\geq 1$, can be replaced by $b_2\neq 0$. Besides, the constants $C^{P^{[1,\nu]}}$, C^Q, and κ are
$$C^{P^{[1,\nu]}}=\alpha_1^{P^{[1,\nu]}}-\frac{\beta_2^{P^{[1,\nu]}}(a_2-b_2)}{b_2(a_1-b_1)}, \quad C^Q=\alpha_1^Q-\frac{\beta_2^Q(a_2-b_2)}{a_2(a_1-b_1)},$$
$$\kappa=\frac{\beta_2^{P^{[1,\nu]}}a_2}{\beta_2^Q b_2}\,\frac{\langle \mathcal{U},\sigma(x;\nu)\rangle}{\langle \mathcal{V},1\rangle}.$$
- In the proof $(ii)\Longrightarrow(i)$, we get $P_1^{[1,\nu]}(x)-Q_1(x)=b_1-a_1\neq 0$, $a_1b_1\neq 0$ and
$$a_n=-\frac{\langle \mathcal{V}, P_n^{[1,\nu]}(x)\rangle}{\langle \mathcal{V}, P_{n-1}^{[1,\nu]}(x)\rangle}\neq 0, \quad b_n=-\frac{\langle \sigma(x;\nu)\mathcal{U}, Q_n(x)\rangle}{\langle \sigma(x;\nu)\mathcal{U}, Q_{n-1}(x)\rangle}\neq 0, \quad n\geq 2.$$

- From the proof of $(ii) \Rightarrow (i)$ we get

$$C^{P^{[1,\nu]}} = \alpha_{n+1}^{Q} - b_{n+1} - \frac{\beta_{n+1}^{Q}}{b_n}, \quad C^{Q} = \alpha_{n+1}^{P^{[1,\nu]}} - a_{n+1} - \frac{\beta_{n+1}^{P^{[1,\nu]}}}{a_n}, \quad n \geq 2.$$

Finally, the following corollary is a straightforward consequence of Theorem 3.1.15, Theorem 1.5.3, and Proposition 1.4.8.

Corollary 3.1.17. *Let $\mathcal{U}$ be a D_ν-classical linear functional given by (3.1.43), let $\mathcal{V}$ be a regular linear functional, and let $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ be their corresponding SMOP. The following statements are equivalent*

i) *$(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ν-coherent pair given by (3.1.1), with $a_1 \neq b_1$ and $a_n b_n \neq 0$, for $n \geq 1$.*

ii) *$P_n^{[1,\nu]}(x) \neq Q_n(x)$, for $n \geq 1$, and there exist constants $C^{P^{[1,\nu]}}, C^{Q}$, and κ (see Remark 3.1.16) such that*

$$\left(x - C^{P^{[1,\nu]}}\right)\sigma(x;\nu)\mathcal{U} = \kappa\left(x - C^{Q}\right)\mathcal{V}.$$

Therefore, $\mathcal{V}$ is a D_ν-semiclassical linear functional of class at most 2.

Remark 3.1.18. From the previous Corollary and Remark 3.1.16 it follows that if $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ν-coherent pair given by (3.1.1) with $a_1 \neq b_1$ and $b_2 \neq 0$, and $\mathcal{U}$ is a D_ν-classical linear functional given by (3.1.43), then

$$\mathcal{V} = \kappa^{-1}\left(x - C^{Q}\right)^{-1}\left(x - C^{P^{[1,\nu]}}\right)\sigma(x;\nu)\mathcal{U} + \langle \mathcal{V}, 1\rangle\,\delta_{C^{Q}}.$$

In particular, this equation holds when $\mathcal{U}$ is any of the D_1-classical linear functionals given in the Table 1.5.3, or, it is any of the D_q-classical linear functionals stated in the Tables 1.5.4, 1.5.5, and 1.5.6.

Finally, in the following remark, we analyze the $(1,1)$ and $(1,0)$ - D_ν-coherence cases when $\mathcal{U}$ and $\mathcal{V}$ are weakly quasi-definite linear functionals.

Remark 3.1.19. When $\mathcal{U}$ and $\mathcal{V}$ are weakly quasi-definite linear functionals with corresponding family of MOP $\{P_n(x)\}_{n=0}^{\Upsilon_0}$ and $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$, the definitions of $(1,1)$-D_ν-coherent and $(1,0)$-D_ν-coherent pair are the same as before with following additional restrictions

$$\Upsilon_0 \geq 2, \quad \Upsilon_1 \geq 1 \quad \text{and} \quad (3.1.1) \text{ holds for } \; 1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\}.$$

In this way, all the previous results of this section hold with the following modifications

- The equations in Remark 3.1.1 hold for $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\}$.

- (3.1.3) holds for $2 \leq n+1 \leq m \leq \min\{\Upsilon_0-1, \Upsilon_1\}-1$, and for $n = \min\{\Upsilon_0-1, \Upsilon_1\}-1$ we have that $\left\langle \gamma_n(x;\nu)\mathcal{V}, P_{n+1}^{[1,\nu]}(x) \right\rangle = 0$. Furthermore, $\deg(\gamma_n(x;\nu)) = n$ with $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$.

- (3.1.4), (3.1.5), and (3.1.6) hold for $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$.

- Corollary 3.1.5 holds whenever that $\Upsilon_0 \geq 4$ and $\Upsilon_1 \geq 3$. Besides, (3.1.13) holds for $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$.

- Remark 3.1.6 and Theorem 3.1.7 hold if $\Upsilon_0 \geq 4$ and $\Upsilon_1 \geq 3$.

- Lemma 3.1.8 and Lemma 3.1.9 hold if $\Upsilon_0 \geq 4$ and $\Upsilon_1 \geq 3$, and (3.1.19) and (3.1.20) hold for $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$.

- Theorem 3.1.10 hold if $\Upsilon_0 \geq 4$ and $\Upsilon_1 \geq 3$.

- In Theorem 3.1.12, $2 \leq N \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$, $\Upsilon_0 \geq 4$ and $\Upsilon_1 \geq 3$.

- For the $(1,0)$-D_ν-coherence case analyzed in Remark 3.1.13, in addition to the above modifications, we have to set $1 \leq n \leq m \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$ and $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\} - 1$ instead of $m \geq n \geq 1$ and $n \geq 1$, in the first and second item of that remark.

- In Proposition 3.1.14, let us recall that the $(1,0)$-D_ν-coherence relation holds for $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\}$.

- In Theorem 3.1.15, we have to set the families of MOP $\{T_n(x)\}_{n=0}^{\widetilde{\Upsilon}_0}$ and $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$ instead of the SMOP $\{T_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, $\{a_n\}_{n=1}^{\min\{\widetilde{\Upsilon}_0, \Upsilon_1\}}$ and $\{b_n\}_{n=1}^{\min\{\widetilde{\Upsilon}_0, \Upsilon_1\}}$ instead of $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, and $1 \leq n \leq \min\{\widetilde{\Upsilon}_0, \Upsilon_1\}$ instead of $n \geq 1$. Besides, $\widetilde{\Upsilon}_0 \geq 2$ and $\Upsilon_1 \geq 2$.

- In Remark 3.1.16 and Corollary 3.1.17, we must set the families of MOP $\{P_n(x)\}_{n=0}^{\Upsilon_0}$, $\{P_n^{[1,\nu]}(x)\}_{n=0}^{\Upsilon_0-1}$ and $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$ instead of the SMOP $\{P_n(x)\}_{n\geq 0}$, $\{P_n^{[1,\nu]}(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, and, $1 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\}$ and $2 \leq n \leq \min\{\Upsilon_0 - 1, \Upsilon_1\}$ instead of $n \geq 1$ and $n \geq 2$. Furthermore, we have to use the TTRR given in (1.3.8) instead of (1.3.1). Additionally, $\Upsilon_0 \geq 3$ and $\Upsilon_1 \geq 2$.

- The equation given in Remark 3.1.18 holds when $\mathcal{U}$ is any of the D_1-classical linear functionals stated in the Table 1.5.2.

3.2 (M, N)-D_ν-Coherent Pairs of Order (m, k)

Let us remind that a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-D_ν-*coherent pair of order* (m, k), with fixed $M, N, m, k \in \mathbb{N} \cup \{0\}$, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n^{[k,\nu]}(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}^{[k,\nu]}(x), \quad n \geq 0, \tag{3.2.1}$$

where $a_{i,n}, b_{i,n} \in \mathbb{C}$, $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ if $i > n$. Besides, $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-D_ν-*coherent pair of order* m if it is a (M, N)-D_ν-coherent pair of order $(m, 0)$.

Remark 3.2.1. From Theorem 1.5.3, when $(\mathcal{U}, \mathcal{V})$ is a (M, N)-D_ν-coherent pair of order (m, k) and $\mathcal{U}$ (resp. $\mathcal{V}$) is a D_ν-classical linear functional, then $(\mathcal{V}, \mathcal{U})$ (resp. $(\mathcal{U}, \mathcal{V})$) can be seen as a (N, M) (resp. (M, N)) -D_ν-coherent pair of order k (resp. m).

In the next theorem, we state the D_ν-analogue results obtained in [53, 54, 79], and we generalize the results stated in [14, 15, 17, 67, ?] for $\nu = \omega$, and in [14, 16, 77] for $\nu = q$. Moreover, We give a complete description of the D_ν-semiclassical case in the framework of (M, N)-D_ν-coherence of order (m, k).

Theorem 3.2.2. *Let $(\mathcal{U}, \mathcal{V})$ be a (M, N)-D_ν-coherent pair of order (m, k) given by (3.2.1) with $m \geq k$. Let $\mathcal{L}_{M+N} = [l_{i,j}]_{i,j=0}^{M+N-1}$ be the following matrix of order $M + N$*

$$l_{i,j} = \begin{cases} a_{j-i,j} & \text{if} \quad 0 \leq i \leq N-1 \quad \text{and} \quad i \leq j \leq M+i, \\ b_{j-i+N,j} & \text{if} \quad N \leq i \leq M+N-1 \quad \text{and} \quad i-N \leq j \leq i, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2.2}$$

with $a_{0,j_1} = b_{0,j_2} = 1$, $0 \leq j_1 \leq N-1$, $0 \leq j_2 \leq M-1$. If $\det(\mathcal{L}_{M+N}) \neq 0$, then there exist polynomials $\phi_{M+k+n}(x;\nu)$, $\psi_{N+m+n}(x;\nu)$, $\varphi(x;\nu)$ and $\rho(x;\nu)$, with $\deg(\phi_{M+k+n}(x;\nu)) = M+k+n$ and $\deg(\psi_{N+m+n}(x;\nu)) = N+m+n$, such that

$$D_{\nu^*}^{m-k}[\phi_{M+k+n}(x;\nu)\mathcal{V}] = \psi_{N+m+n}(x;\nu)\mathcal{U}, \quad n \geq 0, \tag{3.2.3}$$
$$\varphi(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V}. \tag{3.2.4}$$

Moreover,

(i) *If $m = k$, then $\mathcal{U}$ is a D_ν-semiclassical linear functional if and only if so is $\mathcal{V}$.*

(ii) *If $m > k$, then $\mathcal{U}$ and $\mathcal{V}$ are D_ν-semiclassical linear functionals.*

Proof. From (3.2.1), let $a_{0,n}=b_{0,n}=1$ and

$$R_n(x;\nu)=\sum_{i=0}^{M}a_{i,n}P_{n-i}^{[m,\nu]}(x)=\sum_{i=0}^{N}b_{i,n}Q_{n-i}^{[k,\nu]}(x),\quad n\geq 0. \tag{3.2.5}$$

Let $\{\mathfrak{p}_n\}_{n\geq 0}$, $\{\mathfrak{q}_n\}_{n\geq 0}$, $\{\mathfrak{r}_{n,\nu}\}_{n\geq 0}$, $\{\mathfrak{e}_{n,m,\nu}\}_{n\geq 0}$, and $\{\mathfrak{h}_{n,k,\nu}\}_{n\geq 0}$ be the dual bases of the SMOP $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and the sequences $\{R_n(x;\nu)\}_{n\geq 0}$, $\{P_n^{[m,\nu]}(x)\}_{n\geq 0}$, and $\{Q_n^{[k,\nu]}(x)\}_{n\geq 0}$, respectively. From

$$\langle\mathfrak{e}_{n,m,\nu},R_j(x;\nu)\rangle\overset{(3.2.5)}{=}\sum_{i=0}^{M}\langle\mathfrak{e}_{n,m,\nu},a_{i,j}P_{j-i}^{[m,\nu]}(x)\rangle=\begin{cases}a_{j-n,j} & \text{if}\quad n\leq j\leq n+M,\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle\mathfrak{h}_{n,k,\nu},R_j(x;\nu)\rangle\overset{(3.2.5)}{=}\sum_{i=0}^{N}\langle\mathfrak{h}_{n,k,\nu},b_{i,j}Q_{j-i}^{[k,\nu]}(x)\rangle=\begin{cases}b_{j-n,j} & \text{if}\quad n\leq j\leq n+N,\\ 0 & \text{otherwise,}\end{cases}$$

it follows that

$$\mathfrak{e}_{n,m,\nu}=\sum_{j\geq 0}\langle\mathfrak{e}_{n,m,\nu},R_j(x;\nu)\rangle\mathfrak{r}_{j,\nu}=\sum_{j=n}^{n+M}a_{j-n,j}\mathfrak{r}_{j,\nu},\quad n\geq 0, \tag{3.2.6}$$

$$\mathfrak{h}_{n,k,\nu}=\sum_{j\geq 0}\langle\mathfrak{h}_{n,k,\nu},R_j(x;\nu)\rangle\mathfrak{r}_{j,\nu}=\sum_{j=n}^{n+N}b_{j-n,j}\mathfrak{r}_{j,\nu},\quad n\geq 0. \tag{3.2.7}$$

Using (3.2.6) and (3.2.7) for $0\leq n\leq N-1$ and $0\leq n\leq M-1$, respectively, we set

$$\mathcal{L}_{M+N}\begin{bmatrix}\mathfrak{r}_{0,\nu}\\ \vdots\\ \mathfrak{r}_{N-1,\nu}\\ \mathfrak{r}_{N,\nu}\\ \vdots\\ \mathfrak{r}_{N+M-1,\nu}\end{bmatrix}=\begin{bmatrix}\mathfrak{e}_{0,m,\nu}\\ \vdots\\ \mathfrak{e}_{N-1,m,\nu}\\ \mathfrak{h}_{0,k,\nu}\\ \vdots\\ \mathfrak{h}_{M-1,k,\nu}\end{bmatrix},$$

where the matrix $\mathcal{L}_{M+N}$ is given by (3.2.2). By assumption $\det(\mathcal{L}_{M+N})\neq 0$, then we can solve this linear system and obtain, for $0\leq i\leq M+N-1$,

$$\mathfrak{r}_{i,\nu}=\alpha_{i,0}\mathfrak{e}_{0,m,\nu}+\cdots+\alpha_{i,N-1}\mathfrak{e}_{N-1,m,\nu}+\alpha_{i,N}\mathfrak{h}_{0,k,\nu}+\cdots+\alpha_{i,N+M-1}\mathfrak{h}_{M-1,k,\nu}, \tag{3.2.8}$$

where $\alpha_{i,j}$, $0\leq j\leq N+M-1$, are some constants. Besides, for every $i\geq 0$, if we multiply (3.2.6) for $n=N+i$ by $b_{N,M+N+i}$, and (3.2.7) for $n=M+i$ by $a_{M,M+N+i}$, and subtracting the resulting equations, we get

$$\begin{aligned} &b_{N,M+N+i}\mathfrak{e}_{N+i,m,\nu} - a_{M,M+N+i}\mathfrak{h}_{M+i,k,\nu} \\ &\qquad = \beta_{1,i}\mathfrak{r}_{\min\{M,N\}+i,\nu} + \cdots + \beta_{\max\{M,N\},i}\mathfrak{r}_{M+N+i-1,\nu}, \quad i \geq 0, \quad (3.2.9) \end{aligned}$$

where $\beta_{j,i}$, $1 \leq j \leq \max\{M,N\}$, $i \geq 0$, are constants. Additionally, for $t \geq 0$ fixed, using (3.2.6) we can recursively obtain an expression for $\mathfrak{r}_{M+N+t,\nu}$ as a linear combination of $\mathfrak{r}_{i,\nu}$, $0 \leq i \leq M+N-1$, and $\mathfrak{e}_{j,m,\nu}$, $N \leq j \leq N+t$, (since $a_{M,M+j} \neq 0$, $N \leq j \leq N+t$). Hence, from (3.2.8), (3.2.9) becomes

$$\begin{aligned} &\tilde{\alpha}_{i,0}\mathfrak{e}_{0,m,\nu} + \cdots + \tilde{\alpha}_{i,N+i-1}\mathfrak{e}_{N+i-1,m,\nu} + b_{N,M+N+i}\mathfrak{e}_{N+i,m,\nu} \\ &\qquad = \tilde{\beta}_{i,0}\mathfrak{h}_{0,k,\nu} + \cdots + \tilde{\beta}_{i,M-1}\mathfrak{h}_{M-1,k,\nu} + a_{M,M+N+i}\mathfrak{h}_{M+i,k,\nu}, \quad i \geq 0, \end{aligned}$$

where $\tilde{\alpha}_{i,j_1}$, $\tilde{\beta}_{i,j_2}$, for $0 \leq j_1 \leq N+i-1$, $0 \leq j_2 \leq M-1$, are constants. Applying the mth D_ν-derivative $D^m_{\nu^*}$ and using (1.3.5), since $m \geq k$, we get

$$\begin{aligned} &\widehat{\alpha}_{i,0}\mathfrak{p}_m + \cdots + \widehat{\alpha}_{i,N+i-1}\mathfrak{p}_{N+i-1+m} + b_{N,M+N+i}(-1)^m\eta_{N+i,m,\nu}\mathfrak{p}_{N+i+m} = \\ &\qquad D^{m-k}_{\nu^*}\left[\widehat{\beta}_{i,0}\mathfrak{q}_k + \cdots + \widehat{\beta}_{i,M-1}\mathfrak{q}_{M-1+k} + a_{M,M+N+i}(-1)^k\eta_{M+i,k,\nu}\mathfrak{q}_{M+i+k}\right], \end{aligned}$$

for $i \geq 0$. Therefore, from (1.3.5) it follows (3.2.3) with

$$\begin{aligned} \phi_{M+k+n}(x;\nu) &= (-1)^k\frac{\eta_{M+n,k,\nu}a_{M,M+N+n}}{\langle \mathcal{V}, Q^2_{M+k+n}(x)\rangle}x^{M+k+n} + \textit{lower degree terms}, \; n \geq 0, \\ \psi_{N+m+n}(x;\nu) &= (-1)^m\frac{\eta_{N+n,m,\nu}b_{N,M+N+n}}{\langle \mathcal{U}, P^2_{N+m+n}(x)\rangle}x^{N+m+n} + \textit{lower degree terms}, \; n \geq 0. \end{aligned}$$

For each $n \geq 0$, when $m = k$, (3.2.3) becomes (3.2.4) with

$$\rho(x;\nu) = \phi_{M+k+n}(x;\nu), \quad \text{and} \quad \varphi(x;\nu) = \psi_{N+m+n}(x;\nu).$$

Thus, from Proposition 1.4.8, *(i)* holds.

On the other hand, (3.2.3) becomes, for $n \geq 0$,

$$\sum_{j=0}^{m-k}\begin{bmatrix} m-k \\ j \end{bmatrix}_{\nu^*}\hbar^j_{\nu^*}\left(D^j_{\nu^*}\phi_{M+k+n}\right)(x\underbrace{\star\nu^*\cdots\star\nu^*}_{m-k-j \text{ times}};\nu)\, D^{m-k-j}_{\nu^*}\mathcal{V} = \psi_{N+m+n}(x;\nu)\mathcal{U}.$$

From these equations for $n = 0, 1, \ldots m-k$, we can consider the linear systems

$$\mathcal{T}_{m-k+1}(x;\nu)\begin{bmatrix} D^{m-k}_{\nu^*}\mathcal{V} \\ \vdots \\ D_{\nu^*}\mathcal{V} \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} \psi_{N+m}(x;\nu)\mathcal{U} \\ \psi_{N+m+1}(x;\nu)\mathcal{U} \\ \vdots \\ \psi_{N+m+(m-k)}(x;\nu)\mathcal{U} \end{bmatrix},$$

where

$$\begin{aligned}\rho(x;\nu) &= \det\big(\mathcal{T}_{m-k+1}(x;\nu)\big)\\ &= \det\left(\left[\left(D^i_{\nu^*}\phi_{M+k+n}\right)(\underbrace{x\star\nu^*\cdots\star\nu^*}_{m-k-i \text{ times}};\nu)\right]_{i,n=0}^{m-k}\right)\prod_{j=0}^{m-k}\left[\begin{array}{c} m-k \\ j \end{array}\right]_{\nu^*}\hbar^j_{\nu^*} \neq 0.\end{aligned}$$

If $m > k$, we can solve these systems for $\mathcal{V}$ and $D_{\nu^*}\mathcal{V}$, and then we get (3.2.4) as well as

$$\rho(x;\nu)D_{\nu^*}\mathcal{V} = \varsigma(x;\nu)\mathcal{U},$$

where $\varphi(x;\nu)$ and $\varsigma(x;\nu)$ are some polynomials. Thus,

$$\begin{aligned}&D_{\nu^*}\left[\varphi(x\star\nu;\nu)\rho(x\star\nu;\nu)\mathcal{V}\right] = \varphi(x;\nu)\varsigma(x;\nu)\mathcal{U} + \hbar_{\nu^*}D_{\nu^*}\left[\varphi(x\star\nu;\nu)\rho(x\star\nu;\nu)\right]\mathcal{V}\\ &\quad = \left\{\varsigma(x;\nu)\rho(x;\nu) + \hbar_{\nu^*}D_{\nu^*}\left[\varphi(x\star\nu;\nu)\rho(x\star\nu;\nu)\right]\right\}\mathcal{V},\\ &D_{\nu^*}\left[\varphi(x;\nu)\rho(x\star\nu;\nu)\mathcal{U}\right] = D_{\nu^*}\left[\rho(x\star\nu;\nu)\rho(x;\nu)\mathcal{V}\right]\\ &\quad = \rho(x;\nu)\rho(x\star\nu^*;\nu)D_{\nu^*}\mathcal{V} + \hbar_{\nu^*}D_{\nu^*}\left[\rho(x\star\nu;\nu)\rho(x;\nu)\right]\mathcal{V}\\ &\quad = \left\{\rho(x\star\nu^*;\nu)\varsigma(x;\nu) + \hbar_{\nu^*}\varphi(x;\nu)D_{\nu^*}\left[\rho(x;\nu) + \rho(x\star\nu;\nu)\right]\right\}\mathcal{U},\end{aligned}$$

i.e., $\mathcal{V}$ and $\mathcal{U}$ are D_{ν^*}-semiclassical linear functionals. Therefore, *(ii)* follows from Proposition 1.4.7. □

Remark 3.2.3. When $(\mathcal{U},\mathcal{V})$ is a (M,N)-D_ν-coherent pair of order (m,k) and $m = k$, we can not conclude that $\mathcal{U}$ and $\mathcal{V}$ are D_ν-semiclassical. In fact, if $\mathcal{U}$ and $\mathcal{V}$ satisfy

$$\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}, \quad \deg(\varphi(x)) = N,\ \deg(\rho(x)) = M,$$

then for $n \geq 0$, (see [105])

$$\sum_{i=n-M}^{n+N} a_{i,n,1}P_i(x) = \sum_{i=n-N}^{n+N} b_{i,n,1}Q_i(x), \qquad \sum_{i=n-M}^{n+M} a_{i,n,2}P_i(x) = \sum_{i=n-N}^{n+M} b_{i,n,2}Q_i(x).$$

Hence, we can choose $\mathcal{U}$ or $\mathcal{V}$ being non D_ν-semiclassical, and then, so is the other one.

We have proved that the D_ν-semiclassical character and relation of rational type for a pair of the regular linear functionals are necessary conditions for its (M,N)-D_ν-coherence. Let us now consider the converse result.

Theorem 3.2.4. *Let $\mathcal{U}$ and $\mathcal{V}$ two D_ν-semiclassical linear functionals related by a rational factor, i.e., there exist monic polynomials $\sigma(x)$ and $\varphi(x)$ and nonzero polynomials $\tau(x)$ and $\rho(x)$, such that*

$$D_{\nu^*}\left[\sigma(x)\mathcal{V}\right] = \tau(x)\mathcal{V}, \quad \text{and} \quad \varphi(x)\mathcal{U} = \rho(x)\mathcal{V},$$

$$\deg(\sigma(x)) = \ell, \quad \deg(\tau(x)) = t \geq 1, \quad \deg(\varphi(x)) = \jmath, \quad \deg(\rho(x)) = r,$$

hold. Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be the SMOP associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Then,

$$\sum_{i=n-r-\ell}^{n+\jmath+\ell} a_{i,n} P_i^{[1,\nu]}(x) = \sum_{i=n-\jmath-s}^{n+\jmath+\ell} b_{i,n} Q_i(x), \tag{3.2.10}$$

where $a_{n+\jmath+\ell,n} b_{n+\jmath+\ell,n} \neq 0$, for $n \geq 0$, and $s = \max\{\ell-2, t-1\}$. Therefore, $(\mathcal{U},\mathcal{V})$ is a $(\jmath+2\ell+r, 2\jmath+\ell+s)$-$D_\nu$-coherent pair.

Proof. From Proposition 1.4.9, it follows that

$$\sigma(x) Q_n^{[1,\nu]}(x) = \sum_{i=n-s}^{n+\ell} \xi_{i,n,1} Q_i(x), \quad n \geq s, \qquad \xi_{n-s,n,1} \neq 0, \ n \geq s+1, \tag{3.2.11}$$

where $s = \max\{\ell-2, t-1\}$. Besides, from $\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}$, for every $n \geq 0$, we get

$$\varphi(x) Q_n(x) = \sum_{i=0}^{n+\jmath} \xi_{i,n,2} P_i(x),$$

where

$$\langle \mathcal{U}, P_i^2(x) \rangle \xi_{i,n,2} = \langle \mathcal{U}, \varphi(x) Q_n(x) P_i(x) \rangle = \langle \rho(x)\mathcal{V}, Q_n(x) P_i(x) \rangle = 0, \quad i + r \leq n-1.$$

Thus,

$$\varphi(x) Q_n(x) = \sum_{i=n-r}^{n+\jmath} \xi_{i,n,2} P_i(x), \quad n \geq r. \tag{3.2.12}$$

Furthermore,

$$\sigma(x \star \nu^*) P_n(x) = \sum_{i=n-\ell}^{n+\ell} \xi_{i,n,3} P_i(x), \quad n \geq \ell, \tag{3.2.13}$$

$$D_\nu \left[\varphi(x)\sigma(x\star\nu^*)\right] Q_{n+1}(x) = \sum_{i=n-\jmath-\ell+2}^{n+\jmath+\ell} \xi_{i,n,4} Q_i(x), \quad n \geq \jmath+\ell-2, \tag{3.2.14}$$

$$\varphi(x \star \nu) Q_n(x) = \sum_{i=n-\jmath}^{n+\jmath} \xi_{i,n,5} Q_i(x), \quad n \geq \jmath, \tag{3.2.15}$$

where

$$\langle \mathcal{U}, P_i^2(x) \rangle \xi_{i,n,3} = \langle \mathcal{U}, \sigma(x\star\nu^*) P_n(x) P_i(x) \rangle,$$

$$\begin{aligned}\left\langle \mathcal{V}, Q_i^2(x) \right\rangle \xi_{i,n,4} &= \left\langle \mathcal{V}, D_\nu\left[\varphi(x)\sigma(x\star\nu^*)\right] Q_{n+1}(x) Q_i(x) \right\rangle,\\ \left\langle \mathcal{V}, Q_i^2(x) \right\rangle \xi_{i,n,5} &= \left\langle \mathcal{V}, \varphi(x\star\nu) Q_n(x) Q_i(x) \right\rangle.\end{aligned}$$

On the other hand,

$$D_\nu\left[\varphi(x)\sigma(x\star\nu^*)Q_{n+1}(x)\right] = D_\nu\left[\varphi(x)\sigma(x\star\nu^*)\right] Q_{n+1}(x) + \varphi(x\star\nu)\sigma(x) D_\nu\left[Q_{n+1}(x)\right]. \tag{3.2.16}$$

Let us compute each term in the previous D_ν-derivative

$$\begin{aligned}D_\nu\left[\varphi(x)\sigma(x\star\nu^*)Q_{n+1}(x)\right] &\overset{(3.2.12)}{\underset{(3.2.13)}{=}} \sum_{i=n+1-r}^{n+1+\jmath} \xi_{i,n+1,2} \sum_{j=i-\ell}^{i+\ell} \xi_{j,i,3} D_\nu\left[P_j(x)\right]\\ &= \sum_{i=n-r-\ell+1}^{n+\jmath+\ell+1} \xi_{i,n,6} \frac{D_\nu\left[P_i(x)\right]}{\eta_{i-1,1,\nu}} = \sum_{i=n-r-\ell}^{n+\jmath+\ell} \xi_{i+1,n,6} P_i^{[1,\nu]}(x),\\ \varphi(x\star\nu)\sigma(x) D_\nu\left[Q_{n+1}(x)\right] &\overset{(3.2.11)}{\underset{(3.2.15)}{=}} \eta_{n,1,\nu} \sum_{i=n-s}^{n+\ell} \xi_{i,n,1} \sum_{j=i-\jmath}^{i+\jmath} \xi_{j,i,5} Q_j(x) \sum_{i=n-s-\jmath}^{n+\ell+\jmath} \xi_{i,n,7} Q_i(x).\end{aligned}$$

Consequently, from (3.2.14) and taking into account that $s \geq \ell - 2$, (3.2.16) becomes (3.2.10). □

Remark 3.2.5. A similar discussion as in Remark 3.1.19 can be done for the results obtained in this section when $(\mathcal{U}, \mathcal{V})$ is a (M,N)-D_ν-coherent pair of order (m,k) of weakly quasi-definite linear functionals. Indeed, this is possible using, for instance, (1.3.9) instead of (1.3.5), and taking into account that, in this case, the (M,N)-D_ν-coherence relation of order (m,k) is given by (3.2.1) with the following additional constraints

$$\begin{gathered}P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n^{[k,\nu]}(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}^{[k,\nu]}(x),\\ 0 \leq n \leq \min\{\Upsilon_0 - m, \Upsilon_1 - k\}, \quad 0 \leq M, m \leq \Upsilon_0, \quad 0 \leq N, k \leq \Upsilon_1,\end{gathered}$$

where $a_{i,n}, b_{i,n} \in \mathbb{C}$, $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ if $i > n$, and $\{P_n(x)\}_{n=0}^{\Upsilon_0}$ and $\{Q_n(x)\}_{n=0}^{\Upsilon_1}$ are the corresponding family of MOP with respect to the weakly quasi-definite linear functionals $\mathcal{U}$ and $\mathcal{V}$ of order Υ_0 and Υ_1, respectively.

3.3 D_ν-Sobolev Orthogonal Polynomials and (M,N)-D_ν-Coherent Pairs of Order m

In this section, $\mathbb{P}$ will denote the linear space of polynomials with real coefficients and for a fixed $m \geq 1$, we will consider the Sobolev inner product

$$\langle p(x), r(x) \rangle_{\lambda,\nu} = \langle \mathcal{U}, p(x)r(x) \rangle + \lambda \left\langle \mathcal{V}, (D_\nu^m p)(x)(D_\nu^m r)(x) \right\rangle, \quad \lambda > 0, \tag{3.3.1}$$

where $\mathcal{U}$ and $\mathcal{V}$ are regular linear functionals (or discrete measures supported on, either a uniform lattice when $\nu=\omega$, or a q-lattice when $\nu=q$). Let $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ be the SMOP with respect to $\mathcal{U}$, $\mathcal{V}$ and $\langle\cdot\,,\cdot\rangle_{\lambda,\nu}$, respectively.

Remark 3.3.1. The following algebraic relations hold

$$Q_n(x) = P_n^{[m,\nu]}(x) + \sum_{j=0}^{n-1} \frac{\eta_{j,m,\nu}}{\eta_{n,m,\nu}} \frac{\langle \mathcal{U}, T_{n+m}(x;\nu)P_{j+m}(x)\rangle}{\langle \mathcal{U}, P_{j+m}^2(x)\rangle} P_j^{[m,\nu]}(x), \quad n\geq 0, \tag{3.3.2}$$

$$S_n(x;\lambda,\nu) + \sum_{i=m}^{n-1} \frac{\langle \mathcal{U}, T_n(x;\nu)S_i(x;\lambda,\nu)\rangle S_i(x;\lambda,\nu)}{\langle S_i(x;\lambda,\nu), S_i(x;\lambda,\nu)\rangle_{\lambda,\nu}} = P_n(x) + \sum_{i=m}^{n-1} \frac{\langle \mathcal{U}, T_n(x;\nu)P_i(x)\rangle P_i(x)}{\langle \mathcal{U}, P_i^2(x)\rangle}, \tag{3.3.3}$$

for $n\geq m$, and $S_n(x;\lambda,\nu)=P_n(x)$ for $n\leq m$, where

$$T_n(x;\nu) = \lim_{\lambda\longrightarrow\infty} S_n(x;\lambda,\nu)\,, \quad n\geq 0. \tag{3.3.4}$$

Proof. From (3.3.1), $\langle P_n(x), x^i\rangle_{\lambda,\nu}=0$, for $i<n<m$, and, thus, $S_n(x;\lambda,\nu)=P_n(x)$ for $n<m$. Also, from the uniqueness of the SMOP with respect to the inner product $\langle\cdot,\cdot\rangle_{\lambda,\nu}$, each $S_n(x;\lambda,\nu)$ can be written as

$$S_n(x;\lambda,\nu) = \frac{\begin{vmatrix} w_{0,0,\nu} & \cdots & w_{0,n-1,\nu} & w_{0,n,\nu} \\ \vdots & \ddots & \vdots & \vdots \\ w_{n-1,0,\nu} & \cdots & w_{n-1,n-1,\nu} & w_{n-1,n,\nu} \\ 1 & \cdots & x^{n-1} & x^n \end{vmatrix}}{\det\left([w_{i,j,\nu}]_{i,j=0}^{n-1}\right)}, \; n\geq 1, \quad S_0(x;\lambda,\nu)=1,$$

where

$$w_{i,j,\nu} = \langle x^i, x^j\rangle_{\lambda,\nu} = u_{i+j} + \lambda\eta_{i-m,m,\nu}\eta_{j-m,m,\nu}v_{(i-m)+(j-m)}, \quad i,j\geq 0.$$

Hence, every coefficient of $S_n(x;\lambda,\nu)$ is a rational function of λ such that its numerator and denominator have the same degree and, as a consequence, there exists the sequence of monic polynomials given by (3.3.4). On the other hand, from (3.3.4) and (3.3.1) we obtain, for $n\geq 0$,

$$\langle \mathcal{U}, T_n(x;\nu)x^i\rangle = 0,\; i<\min\{n,m\}, \quad \langle \mathcal{V}, D_\nu^m[T_n(x;\nu)]\,x^j\rangle = 0,\; j<n-m. \tag{3.3.5}$$

Indeed, for $i<\min\{n,m\}$

$$\langle \mathcal{U}, T_n(x;\nu)x^i\rangle = \lim_{\lambda\longrightarrow\infty}\left[\langle S_n(x;\lambda,\nu), x^i\rangle_{\lambda,\nu} - \lambda\langle \mathcal{V}, D_\nu^m(S_n(x;\lambda,\nu))D_\nu^m(x^i)\rangle\right] = 0,$$

$$\text{and for } i<n,\ \langle \mathcal{V}, D_\nu^m(T_n(x;\nu))D_\nu^m(x^i)\rangle = \lim_{\lambda\longrightarrow\infty}\frac{1}{\lambda}\left[0-\langle \mathcal{U}, S_n(x;\lambda,\nu)x^i\rangle\right]=0.$$

Then, from (3.3.5),

$$\begin{aligned} T_n(x;\nu) &= \sum_{i=0}^{n}\frac{\langle \mathcal{U}, T_n(x;\nu)P_i(x)\rangle}{\langle \mathcal{U}, P_i^2(x)\rangle}P_i(x) = \sum_{j=0}^{n-m}\frac{\langle \mathcal{U}, T_n(x;\nu)P_{j+m}(x)\rangle}{\langle \mathcal{U}, P_{j+m}^2(x)\rangle}P_{j+m}(x), \quad n\geq m,\\ \frac{D_\nu^m[T_{n+m}(x;\nu)]}{\eta_{n,m,\nu}} &= \sum_{i=0}^{n}\frac{\langle \mathcal{V}, Q_i(x)D_\nu^m[T_{n+m}(x;\nu)]/\eta_{n,m,\nu}\rangle}{\langle \mathcal{V}, Q_i^2(x)\rangle}Q_i(x) = Q_n(x), \quad n\geq 0, \end{aligned} \tag{3.3.6}$$

which proves (3.3.2). Finally, for the proof of (3.3.3), from (3.3.1) and (3.3.5) we get

$$\begin{aligned} T_n(x;\nu) &= \sum_{i=0}^{n}\frac{\langle T_n(x;\nu), S_i(x;\lambda,\nu)\rangle_{\lambda,\nu}}{\langle S_i(x;\lambda,\nu), S_i(x;\lambda,\nu)\rangle_{\lambda,\nu}}S_i(x;\lambda,\nu)\\ &= S_n(x;\lambda,\nu) + \sum_{i=m}^{n-1}\frac{\langle \mathcal{U}, T_n(x;\nu)S_i(x;\lambda,\nu)\rangle}{\langle S_i(x;\lambda,\nu), S_i(x;\lambda,\nu)\rangle_{\lambda,\nu}}S_i(x;\lambda,\nu), \quad n\geq 0. \end{aligned}$$

□

Now, we will study the case when $(\mathcal{U},\mathcal{V})$ is a (M,N)-D_ν-coherent pair of order m, i.e, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n}P_{n-i}^{[m,\nu]}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n}Q_{n-i}(x), \quad n\geq 0, \tag{3.3.7}$$

where $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, and $a_{i,n}=b_{i,n}=0$ when $i>n$.

The following Theorem generalizes a well known algebraic property for $(M,0)$-D_ω-coherent and $(1,1)$-D_ν-coherent pairs, in [67, 76, 77], to (M,N)-D_ν-coherent pairs of order m, and it is the D_ν-analog result showed in [53, 55].

Theorem 3.3.2. *Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-D_ν-coherent pair of order m given by (3.3.7), and $K=\max\{M,N\}$. Then, $S_n(x;\lambda,\nu)=P_n(x)$ for $n<m$ and*

$$P_{n+m}(x) + \sum_{i=1}^{M}\frac{\eta_{n,m,\nu}a_{i,n}}{\eta_{n-i,m,\nu}}P_{n-i+m}(x) = S_{n+m}(x;\lambda,\nu) + \sum_{j=1}^{K} c_{j,n,\lambda,\nu}S_{n-j+m}(x;\lambda,\nu), \quad n\geq 0, \tag{3.3.8}$$

where $c_{j,n,\lambda,\nu}=0$ for $n<j\leq K$, and

$$c_{j,n,\lambda,\nu} = \frac{\eta_{n,m,\nu}}{\langle S_{n-j+m}(x;\lambda,\nu), S_{n-j+m}(x;\lambda,\nu)\rangle_{\lambda,\nu}} \left[\sum_{i=j}^{M} \frac{a_{i,n}\langle \mathcal{U}, P_{n-i+m}(x) S_{n-j+m}(x;\lambda,\nu)\rangle}{\eta_{n-i,m,\nu}} \right.$$
$$\left. + \lambda \sum_{i=j}^{N} b_{i,n}\langle \mathcal{V}, Q_{n-i}(x) D_\nu^m[S_{n-j+m}(x;\lambda,\nu)]\rangle \right], \quad 1 \le j \le K. \quad (3.3.9)$$

Besides, for each $n \ge K$,

(i) *if $M > N$ and $a_{M,n} \neq 0$, then $c_{K,n,\lambda,\nu} \neq 0$,*

(ii) *if $M < N$ and $b_{N,n} \neq 0$, then $c_{K,n,\lambda,\nu} \neq 0$,*

(iii) *if $M = N (= K)$ and $a_{M,n} b_{N,n} \neq 0$ then,*

$$c_{K,n,\lambda,\nu} \neq 0 \quad \text{iff} \quad a_{K,n}\langle \mathcal{U}, P^2_{n-K+m}(x)\rangle + \lambda \eta^2_{n-K,m,\nu} b_{K,n}\langle \mathcal{V}, Q^2_{n-K}(x)\rangle \neq 0.$$

Conversely, if there exist constants $\{c_{j,n,\lambda,\nu}\}_{n\ge0}$, $1 \le j \le K$, and $\{a_{i,n}\}_{n\ge0}$, $1 \le i \le M$, with $c_{j,n,\lambda,\nu} = 0$, $n - j + m < 0$, and $a_{i,n} = 0$, $n - i + m < 0$, such that (3.3.8) holds, then $(\mathcal{U}, \mathcal{V})$ is a (M,K)-D_ν-coherent pair of order m given by

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n(x) + \sum_{j=1}^{K} b_{j,n} Q_{n-j}(x), \quad n \ge 0, \quad (3.3.10)$$

(whenever $b_{K,n} \neq 0$ for $n \ge K$), where $b_{j,n} = 0$ for $n < j \le K$, and

$$b_{j,n} = \frac{\left\langle \mathcal{V}, \left(P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x)\right) Q_{n-j}(x)\right\rangle}{\langle \mathcal{V}, Q^2_{n-j}(x)\rangle}, \quad 1 \le j \le \min\{K,n\},\ n \ge 0. \quad (3.3.11)$$

Proof. $S_n(x;\lambda,\nu) = P_n(x)$, $n < m$, follows from $\langle P_n(x), x^i\rangle_{\lambda,\nu} = 0$, $i < n < m$. On the other hand, substituting (3.3.6) in (3.3.7), and then, computing D_ν-antiderivatives m times (this is, a function $F(x)$ is a D_ν-*antiderivative* of a function $f(x)$ if $D_\nu F(x) = f(x)$, [56]), we obtain

$$\frac{P_{n+m}(x)}{\eta_{n,m,\nu}} + \sum_{i=1}^{M} a_{i,n} \frac{P_{n-i+m}(x)}{\eta_{n-i,m,\nu}} = \frac{T_{n+m}(x;\nu)}{\eta_{n,m,\nu}} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x;\nu)}{\eta_{n-i,m,\nu}} + \sum_{j=0}^{m-1} \kappa_{n,j} x^j, \quad n \ge 0.$$

Taking $\langle x^i \mathcal{U}, \cdot\rangle$, $i < m$, from (3.3.5), we get the linear system

$$\sum_{j=0}^{m-1} \kappa_{n,j} u_{j+i} = 0, \quad i = 0, \dots, m-1.$$

Since $\det\left([u_{i+j}]_{i,j=0}^{m-1}\right) \neq 0$, then $\kappa_{n,j} = 0$, $j = 0, \dots, m-1$, $n \geq 0$. Hence

$$\frac{P_{n+m}(x)}{\eta_{n,m,\nu}} + \sum_{i=1}^{M} a_{i,n} \frac{P_{n-i+m}(x)}{\eta_{n-i,m,\nu}} = \frac{T_{n+m}(x;\nu)}{\eta_{n,m,\nu}} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x;\nu)}{\eta_{n-i,m,\nu}}, \quad n \geq 0. \tag{3.3.12}$$

Furthermore, for $n \geq 0$,

$$\frac{T_{n+m}(x;\nu)}{\eta_{n,m,\nu}} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x;\nu)}{\eta_{n-i,m,\nu}} = \frac{S_{n+m}(x;\lambda,\nu)}{\eta_{n,m,\nu}} + \sum_{j=1}^{n+m} \frac{c_{j,n,\lambda,\nu}}{\eta_{n,m,\nu}} S_{n-j+m}(x;\lambda,\nu),$$

where from (3.3.1), (3.3.12) and (3.3.6), for $1 \leq j \leq n+m$,

$$\begin{aligned}&\langle S_{n-j+m}(x;\lambda,\nu), S_{n-j+m}(x;\lambda,\nu)\rangle_{\lambda,\nu} \frac{c_{j,n,\lambda,\nu}}{\eta_{n,m,\nu}} = \\ &\sum_{i=1}^{M} \frac{a_{i,n} \langle \mathcal{U}, P_{n-i+m}(x) S_{n-j+m}(x;\lambda,\nu)\rangle}{\eta_{n-i,m,\nu}} + \lambda \sum_{i=1}^{N} b_{i,n} \langle \mathcal{V}, Q_{n-i}(x) D_\nu^m[S_{n-j+m}(x;\lambda,\nu)]\rangle,\end{aligned}$$

then $c_{j,n,\lambda,\nu} = 0$ for $j > i$ or $j > K$. Thus, (3.3.8) and (3.3.9) hold. Also, for $n \geq K$,

$$\frac{c_{K,n,\lambda,\nu}}{\eta_{n,m,\nu}} = \frac{\frac{a_{M,n}}{\eta_{n-M,m,\nu}} \langle \mathcal{U}, P^2_{n-M+m}(x)\rangle \delta_{M,K} + \lambda \eta_{n-N,m,\nu} b_{N,n} \langle \mathcal{V}, Q^2_{n-N}(x)\rangle \delta_{N,K}}{\langle S_{n-K+m}(x;\lambda,\nu), S_{n-K+m}(x;\lambda,\nu)\rangle_{\lambda,\nu}},$$

and, as a consequence, *(i)*, *(ii)*, and *(iii)* hold. Finally,

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n(x) + \sum_{j=1}^{n} b_{j,n} Q_{n-j}(x), \quad n \geq 0,$$

with $b_{j,n}$, $1 \leq j \leq n$, given by (3.3.11). Applying $\langle \cdot\,, p(x)\rangle_{\lambda,\nu}$ to both sides of (3.3.8), for $p \in \mathbb{P}_{n-K+m-1}$ we get

$$0 = \lambda \left\langle \mathcal{V}, \left(D_\nu^m[P_{n+m}(x)] + \sum_{i=1}^{M} \frac{\eta_{n,m,\nu} a_{i,n}}{\eta_{n-i,m,\nu}} D_\nu^m[P_{n-i+m}(x)] \right) D_\nu^m[p(x)] \right\rangle,$$

i.e.,

$$0 = \left\langle \mathcal{V}, \left(P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) \right) r(x) \right\rangle, \quad \forall\, r \in \mathbb{P}_{n-K-1}.$$

Thus $b_{j,n} = 0$, for $n - j \leq n - (K+1)$, which proves (3.3.10). $\square$

Remark 3.3.3. When $(\mathcal{U},\mathcal{V})$ is a (M,N)-D_ν-coherent pair of order m, Theorem 3.3.2 allows to compute recursively the D_ν-Sobolev SMOP $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ and the coefficients $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1\leq j\leq K$. Moreover, Theorem 3.3.4 gives a recursive equation for computing the sequences $\{\langle S_n(x;\lambda,\nu),S_n(x;\lambda,\nu)\rangle_{\lambda,\nu}\}_{n\geq 0}$ and $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1\leq j\leq K$, and, thus, using (3.3.8) and $S_n(x;\lambda,\nu)=P_n(x)$ for $n<m$, we can get the D_ν-Sobolev SMOP $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$.

Theorem 3.3.4. *Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-D_ν-coherent pair of order m given by (3.3.7), $K=\max\{M,N\}$, and*

$$s_{n,\nu}=\langle S_n(x;\lambda,\nu),S_n(x;\lambda,\nu)\rangle_{\lambda,\nu},\quad \widetilde{a}_{i,n,\nu}=\frac{\eta_{n,m,\nu}}{\eta_{n-i,m,\nu}}a_{i,n},\quad \widetilde{b}_{i,n,\nu}=\eta_{n,m,\nu}b_{i,n},\quad n\geq 0,$$

with $a_{i,n}=b_{i,n}=0$ if $i>n$, and, $a_{0,n}=b_{0,n}=1$ for $n\geq 0$. Then

$$s_{n+m,\nu}c_{j,n+j,\lambda,\nu}=\zeta_{j,n,\lambda,\nu}-\sum_{\ell=1}^{K-j}c_{\ell,n,\lambda,\nu}c_{j+\ell,n+j,\lambda,\nu}s_{n-\ell+m,\nu},\quad 0\leq j\leq K,\ n\geq 0,\tag{3.3.13}$$

with $s_{n,\nu}=\langle\mathcal{U},P_n^2(x)\rangle$ for $n<m$, $c_{0,n,\lambda,\nu}=1$ for $n\geq 0$, $c_{j,n,\lambda,\nu}=0$ for $n<j\leq K$, and for $0\leq j\leq K$

$$\zeta_{j,n,\lambda,\nu}=\sum_{i=j}^{M}\widetilde{a}_{i,n+j,\nu}\widetilde{a}_{i-j,n,\nu}\langle\mathcal{U},P^2_{n+j-i+m}(x)\rangle+\lambda\sum_{i=j}^{N}\widetilde{b}_{i,n+j,\nu}\widetilde{b}_{i-j,n,\nu}\langle\mathcal{V},Q^2_{n+j-i}(x)\rangle.$$

Proof. Notice that (3.3.8) and (3.3.9) hold setting $c_{0,n,\lambda,\nu}=1$ for $n\geq 0$. Then, from (3.3.7) and (3.3.8), (3.3.9) becomes, for $n\geq j$ and $0\leq j\leq K$,

$$\begin{aligned}s_{n-j+m,\nu}c_{j,n,\lambda,\nu}=&\sum_{i=j}^{M}\sum_{\ell=0}^{M}\widetilde{a}_{i,n,\nu}\widetilde{a}_{\ell,n-j,\nu}\left\langle\mathcal{U},P_{n-i+m}(x)P_{n-j-\ell+m}(x)\right\rangle\\&-\sum_{i=j}^{M}\sum_{\ell=1}^{K}\widetilde{a}_{i,n,\nu}c_{\ell,n-j,\lambda,\nu}\left\langle\mathcal{U},P_{n-i+m}(x)S_{n-j-\ell+m}(x;\lambda,\nu)\right\rangle\\&+\lambda\sum_{i=j}^{N}\sum_{\ell=0}^{N}\widetilde{b}_{i,n,\nu}\widetilde{b}_{\ell,n-j,\nu}\left\langle\mathcal{V},Q_{n-i}(x)Q_{n-j-\ell}(x)\right\rangle\\&-\lambda\sum_{i=j}^{N}\sum_{\ell=1}^{K}\widetilde{b}_{i,n,\nu}c_{\ell,n-j,\lambda,\nu}\left\langle\mathcal{V},Q_{n-i}(x)D_\nu^m[S_{n-j-\ell+m}(x;\lambda,\nu)]\right\rangle.\end{aligned}$$

Since, for $i<j+\ell$ or $j+\ell>K\,(\geq M,N)$

$$\left\langle\mathcal{U},P_{n-i+m}(x)S_{n-j-\ell+m}(x;\lambda,\nu)\right\rangle=0\quad\text{and}\quad\left\langle\mathcal{V},Q_{n-i}(x)D_\nu^m[S_{n-j-\ell+m}(x;\lambda,\nu)]\right\rangle=0,$$

thus, for $n \geq j$ and $0 \leq j \leq K$,

$$\begin{aligned} s_{n-j+m,\nu} c_{j,n,\lambda,\nu} &= \sum_{i=j}^{M} \widetilde{a}_{i,n,\nu} \widetilde{a}_{i-j,n-j,\nu} \langle \mathcal{U}, P^2_{n-i+m}(x) \rangle + \lambda \sum_{i=j}^{N} \widetilde{b}_{i,n,\nu} \widetilde{b}_{i-j,n-j,\nu} \langle \mathcal{V}, Q^2_{n-i}(x) \rangle \\ &- \sum_{\ell=1}^{K-j} c_{\ell,n-j,\lambda,\nu} \sum_{i=j+\ell}^{M} \widetilde{a}_{i,n,\nu} \langle \mathcal{U}, P_{n-i+m}(x) S_{n-j-\ell+m}(x;\lambda,\nu) \rangle \\ &- \lambda \sum_{\ell=1}^{K-j} c_{\ell,n-j,\lambda,\nu} \sum_{i=j+\ell}^{N} \widetilde{b}_{i,n,\nu} \langle \mathcal{V}, Q_{n-i}(x) D^m_\nu [S_{n-j-\ell+m}(x;\lambda,\nu)] \rangle, \end{aligned}$$

and from (3.3.9), the sum of the last two terms is $-\sum_{\ell=1}^{K-j} c_{\ell,n-j,\lambda,\nu} s_{n-j-\ell+m,\nu} c_{j+\ell,n,\lambda,\nu}$. Finally, substituting n by $n+j$, we get (3.3.13). $\square$

Remark 3.3.5.

- Using (3.3.13), the computation of the sequences $\{s_{m+n,\nu}\}_{n\geq 0}$ and $\{c_{j,n+j,\lambda,\nu}\}_{n\geq 0}$, $1 \leq j \leq K$, is done successively along the decreasing diagonals of the following matrix with $K+1$ rows and infinitely many columns

$$\begin{bmatrix} s_{m,\nu} & s_{m+1,\nu} & s_{m+2,\nu} & \cdots & & & & \\ 0 & c_{1,1,\lambda,\nu} & c_{1,2,\lambda,\nu} & c_{1,3,\lambda,\nu} & \cdots & & & \\ 0 & 0 & c_{2,2,\lambda,\nu} & c_{2,3,\lambda,\nu} & c_{2,4,\lambda,\nu} & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & c_{K,K,\lambda,\nu} & c_{K,K+1,\lambda,\nu} & c_{K,K+2,\lambda,\nu} & \cdots \end{bmatrix}$$

- From (3.3.13) for $j=0$, the sequence $\{s_{n,\nu}\}_{n\geq 0}$ satisfies the non-homogeneous linear difference equation of order K

$$s_{n+m,\nu} + \sum_{\ell=1}^{K} c^2_{\ell,n,\lambda,\nu} s_{n-\ell+m,\nu} = \zeta_{0,n,\lambda,\nu}, \quad n \geq 0,$$

where $s_{n,\nu} = \langle \mathcal{U}, P^2_n(x) \rangle$ for $n < m$, $c_{j,n,\lambda,\nu} = 0$ for $n < j \leq K$, and

$$\zeta_{0,n,\lambda,\nu} = \sum_{i=0}^{M} \widetilde{a}^2_{i,n,\nu} \langle \mathcal{U}, P^2_{n-i+m}(x) \rangle + \lambda \sum_{i=0}^{N} \widetilde{b}^2_{i,n,\nu} \langle \mathcal{V}, Q^2_{n-i}(x) \rangle, \quad n \geq 0.$$

3.3.1 Two Particular Cases

When $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ν-coherent or $(1,0)$-D_ν-coherent pair of order m, the sequences $\{s_{n,\nu} = \langle S_n(x;\lambda,\nu), S_n(x;\lambda,\nu) \rangle_{\lambda,\nu}\}_{n\geq 0}$ and $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1 \leq j \leq K$, appearing in (3.3.8), (3.3.9) and (3.3.13) satisfy some additional properties.

Theorem 3.3.6. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ν-coherent pair of order m given by*

$$P_n^{[m,\nu]}(x)+a_{1,n}P_{n-1}^{[m,\nu]}(x)=Q_n(x)+b_{1,n}Q_{n-1}(x),\quad n\geq 0,$$

where $a_{1,0}=b_{1,0}=0$. Let $s_{n,\nu}=\langle S_n(x;\lambda,\nu),S_n(x;\lambda,\nu)\rangle_{\lambda,\nu}$, $n\geq 0$, where $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ is the D_ν-Sobolev SMOP with respect to the inner product (3.3.1). Then,

(i) *The SMOP $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ satisfies $S_n(x;\lambda,\nu)=P_n(x)$ for $n<m$, and*

$$P_{m+n}(x)+\frac{\eta_{n,m,\nu}a_{1,n}}{\eta_{n-1,m,\nu}}P_{m+n-1}(x)=S_{m+n}(x;\lambda,\nu)+c_{1,n,\lambda,\nu}S_{m+n-1}(x;\lambda,\nu),\ n\geq 0, \tag{3.3.14}$$

where $c_{1,0,\lambda,\nu}=0$ and

$$c_{1,n,\lambda,\nu}s_{m+n-1}=\frac{\eta_{n,m,\nu}}{\eta_{n-1,m,\nu}}a_{1,n}\langle\mathcal{U},P_{m+n-1}^2(x)\rangle+\lambda\eta_{n-1,m,\nu}\eta_{n,m,\nu}b_{1,n}\langle\mathcal{V},Q_{n-1}^2(x)\rangle.$$

(ii) *The sequences $\{s_{n,\nu}\}_{n\geq 0}$ and $\{c_{1,n,\lambda,\nu}\}_{n\geq 0}$ in (3.3.14) can be computed by*

$$s_{m+n+1,\nu}=\zeta_{0,n+1,\lambda,\nu}-\frac{\zeta_{1,n,\lambda,\nu}^2}{s_{m+n,\nu}},\qquad \frac{\zeta_{1,n,\lambda,\nu}}{c_{1,n+1,\lambda,\nu}}=\zeta_{0,n,\lambda,\nu}-\zeta_{1,n-1,\lambda,\nu}c_{1,n,\lambda,\nu},\quad n\geq 0, \tag{3.3.15}$$

(whenever $\zeta_{1,n,\lambda,\nu}\neq 0$ for $n\geq 0$), and

$$c_{1,n+1,\lambda,\nu}=\frac{\zeta_{1,n,\lambda,\nu}}{s_{n+m,\nu}},\quad n\geq 0,$$

with initial conditions $c_{1,0,\lambda,\nu}=0$, $s_{m,\nu}=\zeta_{0,0,\lambda,\nu}$, and $s_{n,\nu}=\langle\mathcal{U},P_n^2(x)\rangle$, $n<m$, where

$$\zeta_{1,n,\lambda,\nu}=\frac{\eta_{n+1,m,\nu}}{\eta_{n,m,\nu}}a_{1,n+1}\langle\mathcal{U},P_{m+n}^2(x)\rangle+\lambda\eta_{n,m,\nu}\eta_{n+1,m,\nu}b_{1,n+1}\langle\mathcal{V},Q_n^2(x)\rangle,$$

$$\begin{aligned}\zeta_{0,n,\lambda,\nu}=\langle\mathcal{U},P_{n+m}^2(x)\rangle&+\frac{\eta_{n,m,\nu}^2}{\eta_{n-1,m,\nu}^2}a_{1,n}^2\langle\mathcal{U},P_{n+m-1}^2(x)\rangle\\&+\lambda\eta_{n,m,\nu}^2\left[\langle\mathcal{V},Q_n^2(x)\rangle+b_{1,n}^2\langle\mathcal{V},Q_{n-1}^2(x)\rangle\right].\end{aligned}$$

(iii) *If $\zeta_{1,n,\lambda,\nu}\neq 0$ for $n\geq 0$, then $s_{m+n,\nu}$, $n\geq 0$, and $c_{1,n,\lambda,\nu}$, $n\geq 1$, in (3.3.14), can be represented by continued fractions as*

$$s_{m+n,\nu}=\frac{\zeta_{1,n,\lambda,\nu}^2}{|\zeta_{0,n+1,\lambda,\nu}}-\frac{\zeta_{1,n+1,\lambda,\nu}^2|}{|\zeta_{0,n+2,\lambda,\nu}}-\cdots,\quad n\geq 0,$$

$$c_{1,n,\lambda,\nu}=\frac{\zeta_{0,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}}-\frac{\frac{\zeta_{1,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}}}{\left|\frac{\zeta_{0,n+1,\lambda,\nu}}{\zeta_{1,n,\lambda,\nu}}\right.}-\frac{\left.\frac{\zeta_{1,n+1,\lambda,\nu}}{\zeta_{1,n,\lambda,\nu}}\right|}{\left|\frac{\zeta_{0,n+2,\lambda,\nu}}{\zeta_{1,n+1,\lambda,\nu}}\right.}-\cdots,\quad n\geq 1.$$

(iv) *If $\zeta_{1,n,\lambda,\nu} \neq 0$ for $n \geq 0$, then $\{s_{n,\nu}\}_{n\geq 0}$ and $\{c_{1,n,\lambda,\nu}\}_{n\geq 0}$ in (3.3.14) satisfy*

$$s_{m+n,\nu} = \frac{\varpi_{n+1}(0;\lambda,\nu)}{\varpi_n(0;\lambda,\nu)}, \quad c_{1,n+1,\lambda,\nu} = \zeta_{1,n,\lambda,\nu}\frac{\varpi_n(0;\lambda,\nu)}{\varpi_{n+1}(0;\lambda,\nu)}, \quad n \geq 0, \tag{3.3.16}$$

where $\{\varpi_n(x;\lambda,\nu)\}_{n\geq 0}$ is a SMOP that, besides, will be orthogonal with respect to some positive definite linear functional when $\zeta_{0,n,\lambda,\nu}, \zeta_{1,n,\lambda,\nu} \in \mathbb{R}$ for $n \geq 0$. Moreover, it satisfies the TTRR: $\varpi_0(x;\lambda,\nu) = 1$, $\varpi_{-1}(x;\lambda,\nu) = 0$,

$$\varpi_{n+1}(x;\lambda,\nu) = (x + \zeta_{0,n,\lambda,\nu})\varpi_n(x;\lambda,\nu) - \zeta^2_{1,n-1,\lambda,\nu}\varpi_{n-1}(x;\lambda,\nu), \quad n \geq 0. \tag{3.3.17}$$

Proof. ***(i)*** It follows in a straightforward way from Theorem 3.3.2.
(ii) Since (3.3.13) becomes

$$s_{n+m,\nu} = \zeta_{0,n,\lambda,\nu} - c^2_{1,n,\lambda,\nu}s_{n+m-1,\nu} \quad \text{and} \quad s_{n+m,\nu}c_{1,n+1,\lambda,\nu} = \zeta_{1,n,\lambda,\nu}, \quad n \geq 0,$$

then (3.3.15) holds.
(iii) (3.3.15) becomes, $c_{1,1,\lambda,\nu} = \frac{\zeta_{1,0,\lambda,\nu}}{\zeta_{0,0,\lambda,\nu}}$,

$$c_{1,n,\lambda,\nu} = \frac{\zeta_{0,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}} - \frac{\frac{\zeta_{1,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}}}{c_{1,n+1,\lambda,\nu}}, \; n \geq 1, \quad s_{m+n,\nu} = \frac{\zeta^2_{1,n,\lambda,\nu}}{\zeta_{0,n+1,\lambda,\nu} - s_{m+n+1,\nu}}, \; n \geq 0.$$

(iv) Using the theory of continued fractions, let $\{\varpi_{n,\lambda,\nu}\}_{n\geq 0}$ be the sequence defined by

$$\varpi_{0,\lambda,\nu} = 1 \quad \text{and} \quad \varpi_{n+1,\lambda,\nu} = s_{m+n,\nu}\varpi_{n,\lambda,\nu}, \; n \geq 0.$$

Then, the first equation in (3.3.15) becomes

$$\varpi_{n+2,\lambda,\nu} = \zeta_{0,n+1,\lambda,\nu}\varpi_{n+1,\lambda,\nu} - \zeta^2_{1,n,\lambda,\nu}\varpi_{n,\lambda,\nu}, \; n \geq 0, \quad \varpi_{1,\lambda,\nu} = \zeta_{0,0,\lambda,\nu}, \quad \varpi_{0,\lambda,\nu} = 1.$$

Since $\zeta_{1,n,\lambda,\nu} \neq 0$ for $n \geq 0$, from Favard Theorem there exists a SMOP $\{\varpi_n(x;\lambda,\nu)\}_{n\geq 0}$ such that $\varpi_n(0;\lambda,\nu) = \varpi_{n,\lambda,\nu}$ for $n \geq 0$. Finally, since $\varpi_{n,\lambda,\nu} \neq 0$ for $n \geq 0$, then (3.3.16) follows. □

Remark 3.3.7. If $\zeta_{1,n,\lambda,\nu} \neq 0$ for $n \geq 0$, we can use similar arguments to those in Theorem 3.3.6.(iv)., for the second equation in (3.3.15). In fact, we can set

$$\theta_{n+1,\lambda,\nu} = \frac{\zeta_{1,n,\lambda,\nu}/\zeta_{1,n-1,\lambda,\nu}}{c_{1,n+1,\lambda,\nu}}\theta_{n,\lambda,\nu}, \; n \geq 1, \quad \theta_{1,\lambda,\nu} = \frac{\zeta_{1,0,\lambda,\nu}}{c_{1,1,\lambda,\nu}}\theta_{0,\lambda,\nu}, \quad \theta_{0,\lambda,\nu} = 1.$$

Then (3.3.15) becomes

$$\theta_{0,\lambda,\nu} = 1, \quad \theta_{1,\lambda,\nu} = \zeta_{0,0,\lambda,\nu}, \quad \theta_{2,\lambda,\nu} = \frac{\zeta_{0,1,\lambda,\nu}}{\zeta_{1,0,\lambda,\nu}}\theta_{1,\lambda,\nu} - \zeta_{1,0,\lambda,\nu}\theta_{0,\lambda,\nu},$$

$$\theta_{n+1,\lambda,\nu}=\frac{\zeta_{0,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}}\theta_{n,\lambda,\nu}-\frac{\zeta_{1,n-1,\lambda,\nu}}{\zeta_{1,n-2,\lambda,\nu}}\theta_{n-1,\lambda,\nu},\quad n\geq 2.$$

According to Favard Theorem there exists a SMOP $\{\theta_n(x;\lambda,\nu)\}_{n\geq 0}$ which satisfies the TTRR

$$\begin{aligned}\theta_{n+1}(x;\lambda,\nu)&=\left(x+\frac{\zeta_{0,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}}\right)\theta_n(x;\lambda,\nu)-\frac{\zeta_{1,n-1,\lambda,\nu}}{\zeta_{1,n-2,\lambda,\nu}}\theta_{n-1}(x;\lambda,\nu),\quad n\geq 2,\\ \theta_2(x;\lambda,\nu)&=\left(x+\frac{\zeta_{0,1,\lambda,\nu}}{\zeta_{1,0,\lambda,\nu}}\right)\theta_1(x;\lambda,\nu)-\zeta_{1,0,\lambda,\nu}\theta_0(x;\lambda,\nu),\\ \theta_1(x;\lambda,\nu)&=x+\zeta_{0,0,\lambda,\nu},\quad \theta_0(x;\lambda,\nu)=1,\end{aligned}$$

and $\theta_n(0;\lambda,\nu)=\theta_{n,\lambda,\nu}$ for $n\geq 0$. Furthermore, if $\zeta_{0,n,\lambda,\nu},\zeta_{1,n,\lambda,\nu}\in\mathbb{R}$ and $\zeta_{1,n,\lambda,\nu}>0$ for $n\geq 0$, then $\{\theta_n(x;\lambda,\nu)\}_{n\geq 0}$ is orthogonal with respect to some positive definite linear functional. Additionally, since $\theta_{n,\lambda,\nu}\neq 0$ for $n\geq 0$, then $c_{1,1,\lambda,\nu}=\zeta_{1,0,\lambda,\nu}\frac{\theta_0(0;\lambda,\nu)}{\theta_1(0;\lambda,\nu)}$ and

$$c_{1,n+1,\lambda,\nu}=\frac{\zeta_{1,n,\lambda,\nu}}{\zeta_{1,n-1,\lambda,\nu}}\frac{\theta_n(0;\lambda,\nu)}{\theta_{n+1}(0;\lambda,\nu)},\quad s_{m+n,\nu}=\zeta_{1,n-1,\lambda,\nu}\frac{\theta_{n+1}(0;\lambda,\nu)}{\theta_n(0;\lambda,\nu)},\quad n\geq 1.$$

Remark 3.3.8. Theorem 3.3.6 holds if $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-D_ν-coherent pair of order m, i.e., $b_{1,n}=0$, $n\geq 0$. In this case, for every $n\geq 0$, $\zeta_{1,n,\lambda,\nu}$ is a constant and $\zeta_{0,n,\lambda,\nu}$ is a linear function of λ. Thus, from (3.3.17) and by induction on n, $\varpi_n(0;\lambda,\nu)$ is a polynomial in λ of degree n with leading coefficient $\prod_{i=0}^{n-1}\eta^2_{i,m,\nu}\langle\mathcal{V},Q_i^2(x)\rangle$, $n\geq 1$. Hence, (3.3.17) reads

$$\widetilde{\varpi}_{n+1}(\lambda;\nu)=(\lambda+\alpha_{n,\nu})\widetilde{\varpi}_n(\lambda;\nu)-\beta_{n,\nu}\widetilde{\varpi}_{n-1}(\lambda;\nu),\ \ n\geq 0,\quad \widetilde{\varpi}_0(\lambda;\nu)=1,\tag{3.3.18}$$

where $\widetilde{\varpi}_n(\lambda;\nu)$ is the monic polynomial

$$\widetilde{\varpi}_n(\lambda;\nu)=\frac{\varpi_n(0;\lambda,\nu)}{\prod_{i=0}^{n-1}\eta^2_{i,m,\nu}\langle\mathcal{V},Q_i^2(x)\rangle},\quad n\geq 1,$$

and

$$\begin{aligned}\alpha_{0,\nu}&=\frac{\langle\mathcal{U},P_m^2(x)\rangle}{\eta^2_{0,m,\nu}\langle\mathcal{V},Q_0^2(x)\rangle},\quad \alpha_{n,\nu}=\frac{\langle\mathcal{U},P^2_{n+m}(x)\rangle+\frac{\eta^2_{n,m,\nu}}{\eta^2_{n-1,m,\nu}}a^2_{1,n}\langle\mathcal{U},P^2_{n+m-1}(x)\rangle}{\eta^2_{n,m,\nu}\langle\mathcal{V},Q_n^2(x)\rangle},\ n\geq 1,\\ \beta_{0,\nu}&=0,\quad \beta_{n,\nu}=\frac{a^2_{1,n}\langle\mathcal{U},P^2_{m+n-1}(x)\rangle^2}{\eta^4_{n-1,m,\nu}\langle\mathcal{V},Q_n^2(x)\rangle\langle\mathcal{V},Q^2_{n-1}(x)\rangle},\ n\geq 1.\end{aligned}$$

Therefore, the sequences $\{s_{n,\nu}\}_{n\geq 0}$ and $\{c_{1,n,\lambda,\nu}\}_{n\geq 0}$ in (3.3.14) satisfy

$$s_{m+n,\nu}=\kappa_{n,\nu}\frac{\widetilde{\varpi}_{n+1}(\lambda;\nu)}{\widetilde{\varpi}_n(\lambda;\nu)},\ n\geq 0\quad c_{1,n,\lambda,\nu}=\widetilde{\kappa}_{n,\nu}\frac{\widetilde{\varpi}_{n-1}(\lambda;\nu)}{\widetilde{\varpi}_n(\lambda;\nu)},\ n\geq 1,$$

with

$$\kappa_{n,\nu} = \eta^2_{n,m,\nu}\langle \mathcal{V}, Q^2_n(x)\rangle, \quad \widetilde{\kappa}_{n,\nu} = a_{1,n}\,\frac{\eta_{n,m,\nu}}{\eta^3_{n-1,m,\nu}}\,\frac{\langle \mathcal{U}, P^2_{m+n-1}(x)\rangle}{\langle \mathcal{V}, Q^2_{n-1}(x)\rangle},$$

where $\{\widetilde{\varpi}_n(\lambda;\nu)\}_{n\geq 0}$ is a SMOP in λ satisfying the TTRR (3.3.18), whenever $a_{1,n} \neq 0$ for $n \geq 1$, and it is orthogonal with respect to some positive definite linear functional if $\alpha_{n,\nu} \in \mathbb{R}$ and $\beta_{n+1,\nu} > 0$ for $n \geq 0$.

3.4 A Matrix Interpretation of (M, N)-D_ν-Coherence

Firstly, we will state a general result which, in Remark 3.4.2, we will apply to the notion of (M, N)-D_ν-coherent pair.

Proposition 3.4.1. *Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be two SMOP satisfying*

$$D_\nu \mathbf{p}(x) = \mathcal{M}\mathbf{q}(x), \tag{3.4.1}$$

where

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{q}(x) = \begin{bmatrix} Q_0(x), & Q_1(x), & \cdots \end{bmatrix}^T,$$

and $\mathcal{M}$ is a infinite matrix (such that its 0th row is zero since $D_\nu P_0(x) = 0$). Let $\mathcal{J}_P$ and $\mathcal{J}_Q$ be the Jacobi matrices associated with $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, respectively. Then,

$$\mathcal{J}_P\left(\mathcal{J}_P \star \nu^*\mathcal{I}\right)\mathcal{M} - \mathcal{J}_P\mathcal{M}\mathcal{J}_Q - \left(\mathcal{J}_P \star \nu^*\mathcal{I}\right)\mathcal{M}\mathcal{J}_Q + \mathcal{M}\mathcal{J}_Q^2 = 0. \tag{3.4.2}$$

Proof. From (1.3.2), $\mathbf{p}(x)$ and $\mathbf{q}(x)$ satisfy

$$x\mathbf{p}(x) = \mathcal{J}_P\mathbf{p}(x) \quad \text{and} \quad x\mathbf{q}(x) = \mathcal{J}_Q\mathbf{q}(x). \tag{3.4.3}$$

As a consequence,

$$\begin{aligned}\mathcal{M}\mathcal{J}_Q\mathbf{q}(x) + \hbar_{\nu^*}\mathbf{p}(x) &\underset{(3.4.3)}{\overset{(3.4.1)}{=}} xD_\nu\mathbf{p}(x) + \hbar_{\nu^*}\mathbf{p}(x) = D_\nu\left[(x \star \nu^*)\mathbf{p}(x)\right] \\ &\overset{(3.4.3)}{=} \left(\mathcal{J}_P \star \nu^*\mathcal{I}\right)D_\nu\mathbf{p}(x) \overset{(3.4.1)}{=} \left(\mathcal{J}_P \star \nu^*\mathcal{I}\right)\mathcal{M}\mathbf{q}(x),\end{aligned}$$

or equivalently,

$$\mathbf{p}(x) = \hbar_\nu\big[\left(\mathcal{J}_P \star \nu^*\mathcal{I}\right)\mathcal{M} - \mathcal{M}\mathcal{J}_Q\big]\mathbf{q}(x), \tag{3.4.4}$$

where $\mathcal{I}$ is the infinite identity matrix and

$$\mathcal{J}_P \star \nu\mathcal{I} = \begin{cases} \mathcal{J}_P + \nu\mathcal{I}, & \text{if } \nu = \omega, \\ \nu\mathcal{J}_P, & \text{if } \nu = q. \end{cases}$$

Therefore,

$$\hbar_\nu \mathcal{J}_P\left[(\mathcal{J}_P \star \nu^*\mathcal{I})\,\mathcal{M} - \mathcal{M}\mathcal{J}_Q\right]\mathbf{q}(x) \overset{(3.4.4)}{=} \mathcal{J}_P\mathbf{p}(x) \underset{(3.4.3)}{\overset{(3.4.4)}{=}} \hbar_\nu\left[(\mathcal{J}_P \star \nu^*\mathcal{I})\,\mathcal{M} - \mathcal{M}\mathcal{J}_Q\right]\mathcal{J}_Q\mathbf{q}(x),$$

and (3.4.2) follows. $\square$

Remark 3.4.2. If the SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are such that their corresponding regular linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively, constitute a (M,N)-D_ν-coherent pair given by

$$\frac{D_\nu P_{n+1}(x)}{\eta_{n,1,\nu}} + \sum_{i=1}^{M} a_{i,n}\frac{D_\nu P_{n-i+1}(x)}{\eta_{n-i,1,\nu}} = Q_n(x) + \sum_{i=1}^{N} b_{i,n}Q_{n-i}(x), \quad n\geq 0, \tag{3.4.5}$$

with $a_{M,n}\neq 0$ for $n\geq M$, $b_{N,n}\neq 0$ for $n\geq N$, and $a_{i,n}=b_{i,n}=0$ when $i>n$, then

1. We have the following matrix relation

$$\mathcal{A}_{\nu,1}D_\nu\mathbf{p}_1(x) = \mathcal{B}\mathbf{q}(x), \tag{3.4.6}$$

where $\mathbf{q}(x)$ is given as in (3.4.1),

$$\mathbf{p}_1(x) = \begin{bmatrix} P_1(x), & P_2(x), & \cdots \end{bmatrix}^T,$$

and the matrices $\mathcal{A}_{\nu,1}$ and $\mathcal{B}$ are such that

$$\begin{bmatrix} \underbrace{0 \;\cdots\; 0}_{n-M \text{ zeros}} & \frac{a_{M,n}}{\eta_{n-M,1,\nu}} & \cdots & \frac{a_{1,n}}{\eta_{n-1,1,\nu}} & \frac{1}{\eta_{n,1,\nu}} & 0 & \cdots \end{bmatrix},$$

$$\begin{array}{c} \uparrow \\ n\text{th position} \\ \downarrow \end{array}$$

$$\begin{bmatrix} \underbrace{0 \;\cdots\; 0}_{n-N \text{ zeros}} & b_{N,n} & \cdots & b_{1,n} & 1 & 0 & \cdots \end{bmatrix}$$

are their nth rows, $n\geq 1$, and $[1\;0\;\cdots]$ is their 0th row (counting the rows from zero). Notice that $\mathcal{A}_{\nu,1}$ and $\mathcal{B}$ are nonsingular lower triangular matrices with $M+1$ and $N+1$ nonzero diagonals, respectively, such that the entries of their main diagonals are $\frac{1}{\eta_{n,1,\nu}}$, $n\geq 0$, and 1's, respectively.

Therefore, from (3.4.6), Proposition 3.4.1 holds, where the matrix $\mathcal{M}$ is given by adding a zero row to top of the matrix $\mathcal{A}_1^{-1}\mathcal{B}$ as follows

$$\mathcal{M} = \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_{\nu,1}^{-1}\mathcal{B} \end{bmatrix}, \quad \text{with} \quad \mathbf{0} = \begin{bmatrix} 0, & 0, & \cdots \end{bmatrix}.$$

2. The (M,N)-coherence relation (3.4.5) can be written in an alternative matrix form as
$$\mathcal{A}_\nu D_\nu \mathbf{p}(x) = \mathcal{B}\mathbf{q}(x), \tag{3.4.7}$$
where the infinite vectors $\mathbf{p}(x)$ and $\mathbf{q}(x)$ are given as (3.4.1), the infinite matrix $\mathcal{B}$ is given as in (3.4.6), and the infinite matrix $\mathcal{A}_\nu$ is given by: counting the rows from zero, its 0th row is $\left[1 \ \frac{1}{\eta_{0,1,\nu}} \ 0 \ \cdots\right]$, and, for each $n \geq 1$, its nth row is

$$\begin{array}{c} \left[\begin{array}{cccccccc} \underbrace{0 \ \cdots \ 0}_{n-M+1 \text{ zeros}} & \frac{a_{M,n}}{\eta_{n-M,1,\nu}} & \cdots & \frac{a_{2,n}}{\eta_{n-2,1,\nu}} & \frac{a_{1,n}}{\eta_{n-1,1,\nu}} & \frac{1}{\eta_{n,1,\nu}} & 0 & \cdots \end{array}\right]. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n\text{th position} \end{array}$$

Thus, $\mathcal{A}_\nu$ is a lower Hessenberg matrix with $M+1$ nonzero diagonals such that $\frac{1}{\eta_{n,1,\nu}}$, $n \geq 0$, are the entries of its superdiagonal, and, 1 and $\frac{a_{1,n}}{\eta_{n-1,1,\nu}}$, $n \geq 1$, are the entries of its main diagonal. When $\mathcal{A}_\nu$ is a nonsingular matrix, (e.g., when $M = 1$ and $N \geq 0$, $\mathcal{A}_\nu$ becomes a nonsingular upper bidiagonal matrix since $a_{1,n} \neq 0$ for $n \geq 1$), (3.4.7) can be read as (3.4.1) where $\mathcal{M} = \mathcal{A}_\nu^{-1}\mathcal{B}$, and, as a consequence, Proposition 3.4.1 follows, in particular (3.4.2) holds. Let
$$\mathcal{M}_{P,\nu} = \mathcal{A}_\nu \mathcal{J}_P \mathcal{A}_\nu^{-1} \quad \text{and} \quad \mathcal{M}_Q = \mathcal{B}\mathcal{J}_Q\mathcal{B}^{-1},$$
i.e., $\mathcal{M}_{P,\nu}$ (resp. $\mathcal{M}_Q$) and $\mathcal{J}_P$ (resp. $\mathcal{J}_Q$) are similar matrices. If we multiply on the left by $\mathcal{A}_\nu$ and on the right by $\mathcal{B}^{-1}$ to both sides of (3.4.2), we get
$$\begin{aligned} 0 &= \mathcal{M}_{P,\nu}\left(\mathcal{M}_{P,\nu} \star \nu^*\mathcal{I}\right) - \mathcal{M}_{P,\nu}\mathcal{M}_Q - \left(\mathcal{M}_{P,\nu} \star \nu^*\mathcal{I}\right)\mathcal{M}_Q + \mathcal{M}_Q^2 \\ &= \Big[\mathcal{M}_{P,\nu} - \mathcal{M}_Q\Big]\Big[\left(\mathcal{M}_{P,\nu} \star \nu^*\mathcal{I}\right) - \mathcal{M}_Q\Big] - \Big[\left(\mathcal{M}_{P,\nu} \star \nu^*\mathcal{I}\right)\mathcal{M}_Q - \mathcal{M}_Q\left(\mathcal{M}_{P,\nu} \star \nu^*\mathcal{I}\right)\Big] \\ &= \left(\mathcal{M}_Q - \mathcal{M}_{P,\nu}\right)^2_{\nu^*} - \hbar_{\nu^*}\left[\mathcal{M}_{P,\nu}, \mathcal{M}_Q\right], \end{aligned}$$
where $[\mathcal{R},\mathcal{T}]$ is the *commutator* of the matrices $\mathcal{R}$ and $\mathcal{T}$, defined by
$$[\mathcal{R},\mathcal{T}] = \mathcal{R}\mathcal{T} - \mathcal{T}\mathcal{R},$$
and $(x-a)^n_\nu$, for $a \in \mathbb{C}$ and $n \geq 0$, is the D_ν-analogue of $(x-a)^n$ given by
$$(x-a)^0_\nu = 1, \quad (x-a)^n_\nu = (x-a)\big(x-(a\star\nu)\big)\cdots\Big(x-(a\underbrace{\star\nu\cdots\star\nu}_{n-1 \text{ times}})\Big), \ n \geq 1.$$

Therefore,
$$\left(\mathcal{M}_Q - \mathcal{M}_{P,\nu}\right)^2_{\nu^*} = \hbar_{\nu^*}\left[\mathcal{M}_{P,\nu}, \mathcal{M}_Q\right].$$

Finally, when $\mathcal{U}$ is a D_ν-classical linear functional we have the following result.

Proposition 3.4.3. *If $(\mathcal{U}, \mathcal{V})$ is a (M, N)-D_ν-coherent pair of order m, $m \geq 0$, given by*

$$\widehat{\mathcal{A}}_1 \widehat{\mathbf{p}}_\nu(x) = \mathcal{B}\mathbf{q}(x), \tag{3.4.8}$$

where $\mathbf{q}(x)$ and $\mathcal{B}$ are given as in (3.4.6),

$$\widehat{\mathbf{p}}_\nu(x) = \left[\begin{array}{ccc} P_0^{[m,\nu]}(x), & P_1^{[m,\nu]}(x), & \cdots \end{array} \right]^T,$$

and, $\widehat{\mathcal{A}}_1$ is the lower triangular matrix whose entries of its main diagonal are all 1's, and for $n \geq 1$,

$$\left[\begin{array}{cccccc} \underbrace{0 \quad \cdots \quad 0}_{n-M \ zeros} & a_{M,n} & \cdots & a_{1,n} & 1 & 0 \quad \cdots \end{array} \right]$$

is its nth row (counting the rows from zero), with $a_{M,n} \neq 0$, $n \geq M$, and $a_{i,n} = 0$, $i > n$, and such that $\mathcal{U}$ is a D_ν-classical linear functional, then

$$\widehat{\mathcal{A}}_1 \mathcal{J}_{P^{[m,\nu]}} \widehat{\mathcal{A}}_1^{-1} = \mathcal{M}_{P^{[m,\nu]}} = \mathcal{M}_Q = \mathcal{B}\mathcal{J}_Q\mathcal{B}^{-1},$$

i.e., $\mathcal{J}_{P^{[m,\nu]}}$ and $\mathcal{J}_Q$, the monic Jacobi matrices associated with the SMOP $\{P_n^{[m,\nu]}(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ respectively, are similar matrices.

Proof. Since $\{P_n(x)\}_{n\geq 0}$ is a D_ν-classical SMOP, so is $\{P_n^{[m,\nu]}(x)\}_{n\geq 0}$ from Theorem 1.5.3, as a consequence,

$$x\widehat{\mathbf{p}}_\nu(x) = \mathcal{J}_{P^{[m,\nu]}} \widehat{\mathbf{p}}_\nu(x)$$

holds. Besides, since $\widehat{\mathcal{A}}_1$ is a nonsingular matrix as $\mathcal{B}$, it follows that

$$\widehat{\mathcal{A}}_1 \mathcal{J}_{P^{[m,\nu]}} \widehat{\mathcal{A}}_1^{-1} \mathcal{B}\mathbf{q}(x) \overset{(3.4.8)}{=} \widehat{\mathcal{A}}_1 \mathcal{J}_{P^{[m,\nu]}} \widehat{\mathbf{p}}_\nu(x) = x\widehat{\mathcal{A}}_1 \widehat{\mathbf{p}}_\nu(x) \overset{(3.4.8)}{=} x\mathcal{B}\mathbf{q}(x) \overset{(3.4.3)}{=} \mathcal{B}\mathcal{J}_Q\mathbf{q}(x).$$

Therefore, we can get our result. □

Remark 3.4.4. When $\nu = \omega = 1$, Proposition 3.4.3 holds for the Charlier and Meixner D_1-classical SMOP using the information given in the Table 1.5.3. Similarly, for $\nu = q$, Proposition 3.4.3 can be applied to the D_q-classical SMOP stated in the Tables 1.5.4, 1.5.5, and 1.5.6.

3.4.1 A Matrix Interpretation of D_ν-Sobolev Orthogonal Polynomials and (M, N)-D_ν-Coherence of Order m

Let us consider the Sobolev inner product (3.3.1)

$$\langle p(x), r(x) \rangle_{\lambda,\nu} = \langle \mathcal{U}, p(x)r(x) \rangle + \lambda \left\langle \mathcal{V}, \left(D_\nu^m p\right)(x) \left(D_\nu^m r\right)(x) \right\rangle, \quad \lambda > 0, \ m \in \mathbb{Z}^+,$$

$p(x)$, $r(x)$ polynomials with real coefficients, when the regular linear functionals $\mathcal{U}$ and $\mathcal{V}$ (or discrete measures supported on, either a uniform lattice if $\nu = \omega$, or a q-lattice if

$\nu = q$) constitute a (M,N)-D_ν-coherent pair of order m, $m \geq 1$. Then, from Theorem 3.3.2, the (M,N)-D_ν-coherence relation of order m (3.3.7) yields the algebraic relation (3.3.8), i.e.,

$$\begin{gathered} P_{n+m}(x) + \sum_{i=1}^{M} \widetilde{a}_{i,n,\nu} P_{n-i+m}(x) = S_{n+m}(x;\lambda,\nu) + \sum_{j=1}^{K} c_{j,n,\lambda,\nu} S_{n-j+m}(x;\lambda,\nu), \quad n \geq 0, \\ S_n(x;\lambda,\nu) = P_n(x), \quad n \leq m, \end{gathered} \tag{3.4.9}$$

where $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ is the SMOP with respect to $\langle \cdot\,,\cdot \rangle_{\lambda,\nu}$, $K = \max\{M,N\}$, $c_{j,n,\lambda,\nu} = 0$, $n < j \leq K$, and

$$\widetilde{a}_{i,n,\nu} = \frac{\eta_{n,m,\nu}}{\eta_{n-i,m,\nu}} a_{i,n}, \quad n \geq 0.$$

Thus, writing (3.4.9) in a matrix form as

$$\widetilde{\mathcal{A}}_\nu \mathbf{p}(x) = \mathcal{C}_\nu \mathbf{s}(x;\lambda,\nu), \tag{3.4.10}$$

where the infinite vectors $\mathbf{p}(x)$ and $\mathbf{s}(x;\lambda,\nu)$ are

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{s}(x;\lambda,\nu) = \begin{bmatrix} S_0(x;\lambda,\nu), & S_1(x;\lambda,\nu), & \cdots \end{bmatrix}^T,$$

and the infinite matrices $\widetilde{\mathcal{A}}_\nu$ and $\mathcal{C}_\nu$ are such that

$$\begin{gathered} \begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{n-M+m \text{ zeros}} & \widetilde{a}_{M,n,\nu} & \cdots & \widetilde{a}_{1,n,\nu} & 1 & 0 & \cdots \end{bmatrix} \\ \uparrow \\ (n+m)\text{th place} \\ \downarrow \\ \begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{n-K+m \text{ zeros}} & c_{K,n,\lambda,\nu} & \cdots & c_{1,n,\lambda,\nu} & 1 & 0 & \cdots \end{bmatrix} \end{gathered}$$

are their $(n+m)$th rows, $n \geq 0$, respectively, and $[0\ \cdots\ 0\ 1\ 0\ \cdots]$ are their first m rows (counting from zero), we get the following matrix representation of the multiplication operator by x in terms of the basis of the D_ν-Sobolev orthogonal polynomials $S_n(x;\lambda,\nu)$, $n \geq 0$,

$$x\mathbf{s}(x;\lambda,\nu) = \mathcal{H}_{S,P,\lambda,\nu} \mathbf{s}(x;\lambda,\nu),$$

where $\mathcal{H}_{S,P,\lambda,\nu}$ is a lower Hessenberg matrix similar to the monic Jacobi matrix associated with the SMOP $\{P_n(x)\}_{n\geq 0}$. Indeed,

$$x\mathbf{s}(x;\lambda,\nu) \overset{(3.4.10)}{=} \mathcal{C}_\nu^{-1} \widetilde{\mathcal{A}}_\nu x \mathbf{p}(x) \overset{(1.3.2)}{=} \mathcal{C}_\nu^{-1} \widetilde{\mathcal{A}}_\nu \mathcal{J}_P \mathbf{p}(x) \overset{(3.4.10)}{=} \mathcal{C}_\nu^{-1} \widetilde{\mathcal{A}}_\nu \mathcal{J}_P \widetilde{\mathcal{A}}_\nu^{-1} \mathcal{C}_\nu \mathbf{s}(x;\lambda,\nu),$$

and, as a consequence,

$$\mathcal{H}_{S,P,\lambda,\nu} = \mathcal{C}_\nu^{-1} \widetilde{\mathcal{A}}_\nu \mathcal{J}_P \widetilde{\mathcal{A}}_\nu^{-1} \mathcal{C}_\nu.$$

Besides, notice that the matrices $\widetilde{\mathcal{A}}_\nu$ and $\mathcal{C}_\nu$ are lower triangular matrices which have $M+1$ and $K+1$ nonzero diagonals, respectively, and the entries of their main diagonal are all 1's, and $\mathcal{J}_P$ is the monic Jacobi matrix associated with the linear functional $\mathcal{U}$ which is a tridiagonal matrix.

CHAPTER 4

Coherent Pairs on the Unit Circle

In this chapter we will analyze (M, N)-*coherent pairs of order* m *on the unit circle* for regular Hermitian linear functionals $(\mathcal{U}, \mathcal{V})$ defined on the linear space of Laurent polynomials with complex coefficients, and M, N, m being fixed non-negative integers. This means that their corresponding sequences of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ satisfy

$$\sum_{i=0}^{M} a_{i,n}\phi_{n-i}^{(m)}(z) = \sum_{i=0}^{N} b_{i,n}\psi_{n-i}(z), \quad n \geq 0,$$

where $a_{i,n}$ and $b_{j,n}$, for $0 \leq i \leq M$, $0 \leq j \leq N$, $n \geq 0$, are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and, $a_{i,n} = b_{i,n} = 0$ if $i > n$. When $m = 1$, we will simply say that $(\mathcal{U}, \mathcal{V})$ is a (M, N)-*coherent pair on the unit circle*.

The structure of this chapter is as follows. In Section 4.1, we will describe the $(1, 1)$-coherent pairs $(\mathcal{U}, \mathcal{V})$ when $\mathcal{U}$ and $\mathcal{V}$ are regular Hermitian linear functionals, focusing our attention in the cases when $\mathcal{U}$ is either the Lebesgue or the Bernstein-Szegő linear functional. In Section 4.2, we will consider the Sobolev inner product

$$\left\langle p(z), q(z) \right\rangle_\lambda = \left\langle \mathcal{U}, p(z)\overline{q}(1/z) \right\rangle + \lambda \left\langle \mathcal{V}, p^{(m)}(z)\overline{q^{(m)}}(1/z) \right\rangle, \quad \lambda > 0, \, m \in \mathbb{Z}^+, \tag{4.0.1}$$

and we will study its corresponding sequence of monic OPUC when the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ form a (M, N)-coherent pair of order m. In this way, we will highlight the cases $(M, N) = (1, 1)$ and $(M, N) = (1, 0)$. Finally, in Section 4.3, we will analyze (M, N)-coherent pairs on the unit circle from a matrix point of view, showing an interesting relation for the Hessenberg matrices associated with the regular Hermitian linear functionals that constitute such a (M, N)-coherent pair. As a special case, we will study the case when $\mathcal{U}$ is the linear functional associated with the Lebesgue measure on the unit circle. Furthermore, when $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order m on the unit circle, we will give a matrix representation of the multiplication operator by z in terms of the basis of the Sobolev polynomials orthogonal with respect to (4.0.1), involving a matrix similar to the Hessenberg matrix associated with $\mathcal{U}$.

4.1 $(1,1)$-Coherent Pairs on the Unit Circle

A pair of regular Hermitian linear functionals $(\mathcal{U}, \mathcal{V})$ defined on the linear space of Laurent polynomials is said to be a $(1,1)$-*coherent pair on the unit circle* if their corresponding sequences of monic OPUC, $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$, satisfy the algebraic relation

$$\phi_n^{[1]}(z) + a_n \phi_{n-1}^{[1]}(z) = \psi_n(z) + b_n \psi_{n-1}(z), \quad a_n \neq 0,\ n \geq 1, \tag{4.1.1}$$

where

$$\phi_n^{[1]}(z) = \frac{\phi'_{n+1}(z)}{n+1}, \quad n \geq 0.$$

In such a case, it is also said that $(\{\phi_n(z)\}_{n\geq 0}, \{\psi_n(z)\}_{n\geq 0})$ is a $(1,1)$-coherent pair on the unit circle. If $b_n = 0$ for every $n \geq 1$, then $(\mathcal{U}, \mathcal{V})$ is called a $(1,0)$-*coherent pair on the unit circle.*

Lemma 4.1.1. *If $(\mathcal{U}, \mathcal{V})$ satisfies (4.1.1), then the following statements are equivalent*

i) $a_1 \neq b_1$.

ii) $\phi_n^{[1]}(z) \neq \psi_n(z)$ *for every* $n \geq 1$.

Proof. $(i) \implies (ii)$: If there exists $N \geq 0$, $N \geq 1$, such that $\phi_N^{[1]}(z) = \psi_N(z)$, then from (4.1.1) we get $a_N \phi_{N-1}^{[1]}(z) = b_N \psi_{N-1}(z)$, but since the polynomials are monic, then $a_N = b_N$ and, thus, $\phi_{N-1}^{[1]}(z) = \psi_{N-1}(z)$. If we apply this fact recursively, then

$$a_n = b_n \text{ and } \phi_{n-1}^{[1]}(z) = \psi_{n-1}(z), \text{ for } n = 1, \ldots, N.$$

In particular, $a_1 = b_1$.

$(ii) \implies (i)$: If $a_1 = b_1$ then, from (4.1.1) for $n = 1$, we have $\phi_1^{[1]}(z) = \psi_1(z)$. Thus, $\phi_N^{[1]}(z) = \psi_N(z)$ for $N = 0$ and $N = 1$. ∎

Lemma 4.1.2. *If $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ satisfy (4.1.1), then for $n \geq 1$ we have*

$$\phi_n^{[1]}(z) = \psi_n(z) + (b_n - a_n)\psi_{n-1}(z)$$

$$+\sum_{k=0}^{n-2}(-1)^{n-(k+1)}a_n a_{n-1}\cdots a_{k+2}\left(b_{k+1}-a_{k+1}\right)\psi_k(z), \quad (4.1.2)$$

$$\begin{aligned}\psi_n(z) = \phi_n^{[1]}(z) + (a_n - b_n)\phi_{n-1}^{[1]}(z) \\ + \sum_{k=0}^{n-2}(-1)^{n-(k+1)}b_n b_{n-1}\cdots b_{k+2}\left(a_{k+1}-b_{k+1}\right)\phi_k^{[1]}(z).\end{aligned} \quad (4.1.3)$$

Proof. We will prove (4.1.2) by induction on n. The proof for (4.1.3) is similar. For $n=1$, (4.1.2) is (4.1.1), taking into account that $\psi_0(z) = 1 = \phi_0^{[1]}(z)$. Now, let us assume that (4.1.2) holds for n. Then,

$$\begin{aligned}\phi_{n+1}^{[1]}(z) &\overset{(4.1.1)}{=} \psi_{n+1}(z) + b_{n+1}\psi_n(z) - a_{n+1}\phi_n^{[1]}(z) \\ &\overset{(4.1.2)}{=} \psi_{n+1}(z) + b_{n+1}\psi_n(z) - a_{n+1}\Bigg[\psi_n(z) + (b_n - a_n)\psi_{n-1}(z) \\ &\qquad + \sum_{k=0}^{n-2}(-1)^{n-(k+1)}a_n a_{n-1}\cdots a_{k+2}\left(b_{k+1}-a_{k+1}\right)\psi_k(z)\Bigg] \\ &= \psi_{n+1}(z) + (b_{n+1} - a_{n+1})\psi_n(z) - a_{n+1}(b_n - a_n)\psi_{n-1}(z) \\ &\qquad + \sum_{k=0}^{n-2}(-1)^{n+1-(k+1)}a_{n+1}a_n a_{n-1}\cdots a_{k+2}\left(b_{k+1}-a_{k+1}\right)\psi_k(z),\end{aligned}$$

which is (4.1.2) for $n+1$. □

From the previous Lemma and from the orthogonality of $\{\psi_n(z)\}_{n\geq 0}$ with respect to $\mathcal{V}$, we get the following result.

Corollary 4.1.3. *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-coherent pair on the unit circle given by (4.1.1), then*

$$\left\langle \mathcal{V}, \phi_n^{[1]}(z)\right\rangle = (-1)^n(a_1 - b_1)\prod_{j=2}^{n} a_j \left\langle \mathcal{V}, 1\right\rangle, \quad n \geq 1. \quad (4.1.4)$$

where $\prod_{j=k_1}^{k_2} \cdot = 1$ whenever $k_2 < k_1$.

Next, we will study the $(1,1)$-coherence relation when $\mathcal{U}$ is the linear functional associated with two basic positive measures on the unit circle, namely, the Lebesgue and Bernstein-Szegő measures. For this purpose we will take into account the information stated in the Table 1.6.1.

Notice that, from the results that we will obtain in the following two subsections, we can recover the cases analyzed in [23] and [26] for $(1,0)$ and $(0,1)$ -coherent pairs, respectively.

4.1.1 The Lebesgue Linear Functional

Theorem 4.1.4. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-coherent pair on the unit circle such that their corresponding monic OPUC satisfy (4.1.1) and let $\mathcal{U}$ be the Lebesgue linear functional.*

(i) *If $a_1 = b_1$, then $\mathcal{V}$ is also the linear functional associated with the Lebesgue measure, and*

$$a_n = b_n, \quad n \geq 1.$$

(ii) *If $a_1 \neq b_1$, then the sequence of moments associated with the linear functional $\mathcal{V}$, $\{v_n\}_{n\geq 0}$, is given by*

$$v_n = (-a_2)^{n-1}(b_1 - a_1)v_0, \quad n \geq 1, \tag{4.1.5}$$

and its monic OPUC are

$$\begin{gathered}\psi_1(z) = z + a_1 - b_1, \quad \text{and for } n \geq 2,\\ \psi_n(z) = z^n + (a_2 - b_n)z^{n-1} + \sum_{k=1}^{n-2}(-1)^{n-k-1}b_n \cdots b_{k+2}\,(a_2 - b_{k+1})\,z^k + \beta_n,\end{gathered} \tag{4.1.6}$$

where its associated Verblunsky coefficients $\beta_n = \psi_n(0)$, $n \geq 1$, are such that $|\beta_n| \neq 1$, $n \geq 1$, and

$$\beta_1 = a_1 - b_1, \quad \beta_n = (-1)^{n-1}b_n \cdots b_2\beta_1 = -b_n\beta_{n-1}, \quad n \geq 2. \tag{4.1.7}$$

Furthermore, the coherence coefficients satisfy

$$\begin{gathered}a_2 = b_2(1 - |\beta_1|^2) + \beta_1, \quad a_n = a_2, \quad n \geq 3,\\ b_n = \frac{b_{n-1}}{1 - |\beta_{n-1}|^2} = \frac{b_2}{\prod_{k=2}^{n-1}(1 - |\beta_k|^2)}, \quad n \geq 3.\end{gathered} \tag{4.1.8}$$

Proof. Since, $\phi_n^{[1]}(z) = z^n$ for $n \geq 0$, then (4.1.1) becomes

$$z^n + a_n z^{n-1} = \psi_n(z) + b_n\psi_{n-1}(z), \quad a_n \neq 0,\ n \geq 1. \tag{4.1.9}$$

Thus, applying the linear functional $\mathcal{V}$ in the previous expression we get

$$v_n + a_n v_{n-1} = 0, \quad n \geq 2, \quad \text{and} \quad v_1 + a_1 v_0 = b_1 v_0.$$

As a consequence,

$$\begin{aligned}v_n &= -a_n v_{n-1} = a_n a_{n-1} v_{n-2} = -a_n a_{n-1} a_{n-2} v_{n-3}\\ &= \cdots = (-1)^{n-1}a_n \cdots a_2 v_1 = (-1)^{n-1}a_n \cdots a_2\,(b_1 - a_1)\,v_0, \quad n \geq 2.\end{aligned}$$

Therefore, the moments associated with $\mathcal{V}$ are

$$v_n = -a_n v_{n-1} = (-1)^{n-1} a_n \cdots a_2 \,(b_1 - a_1)\, v_0, \quad n \geq 2, \qquad v_1 = (b_1 - a_1) v_0. \tag{4.1.10}$$

(i) If $a_1 = b_1$, then from (4.1.10) we have $v_n = 0$ for every $n \geq 1$. Thus, $\psi_n(z) = z^n$ for every $n \geq 1$, i.e, $\mathcal{V}$ is also the linear functional associated with the Lebesgue measure, and, as a consequence, from (4.1.9), $a_n = b_n$ for every $n \geq 1$.

(ii) From (4.1.3) we have

$$\psi_n(z) = z^n + (a_n - b_n) z^{n-1} + \sum_{k=0}^{n-2} (-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+2} \,(a_{k+1} - b_{k+1})\, z^k. \tag{4.1.11}$$

Multiplying (4.1.11) by z^{-1}, applying $\mathcal{V}$, and using the orthogonality of $\{\psi_n(z)\}_{n\geq 0}$, we obtain, for n, $n+1$, and $n \geq 2$,

$$\begin{aligned} 0 &= v_{n-1} + (a_n - b_n) v_{n-2} + \sum_{k=0}^{n-2} (-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+2} \,(a_{k+1} - b_{k+1})\, v_{k-1}, \\ 0 &= v_n + (a_{n+1} - b_{n+1}) v_{n-1} \\ &\qquad - b_{n+1}(a_n - b_n) v_{n-2} + \sum_{k=0}^{n-2} (-1)^{n-k} b_{n+1} b_n \cdots b_{k+2} \,(a_{k+1} - b_{k+1})\, v_{k-1}. \end{aligned}$$

Thus, multiplying the first equation by b_{n+1} and adding it to the second equation, we get

$$0 = v_n + a_{n+1} v_{n-1} \overset{(4.1.10)}{=} -a_n v_{n-1} + a_{n+1} v_{n-1} = (a_{n+1} - a_n) v_{n-1}, \quad n \geq 2. \tag{4.1.12}$$

Since $a_n \neq 0$, $n \geq 1$, and $a_1 \neq b_1$, from (4.1.10) we see that $v_n \neq 0$ for $n \geq 1$. Thus, from (4.1.12) we conclude $a_{n+1} = a_n$ for $n \geq 2$ or, equivalently, $a_{n+1} = a_2$ for $n \geq 2$. Therefore, (4.1.10) becomes (4.1.5).

On the other hand, from (4.1.11) we obtain (4.1.7) and (4.1.6). From the forward Szegő relation we have, for $n \geq 0$,

$$\begin{aligned} \psi_{n+1} &= z\psi_n(z) + \beta_{n+1} \psi_n^*(z) \\ &\overset{(4.1.11)}{=} z^{n+1} + (a_n - b_n) z^n + \sum_{k=0}^{n-2} (-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+2} \,(a_{k+1} - b_{k+1})\, z^{k+1} \\ &\qquad + \beta_{n+1} \left[1 + (\overline{a}_n - \overline{b}_n) z + \sum_{k=0}^{n-2} (-1)^{n-(k+1)} \overline{b}_n \overline{b}_{n-1} \cdots \overline{b}_{k+2} \left(\overline{a}_{k+1} - \overline{b}_{k+1}\right) z^{n-k} \right]. \end{aligned}$$

Thus, the comparison of the coefficients of z^n in both expressions yields, for $n \geq 1$,

$$a_{n+1} - b_{n+1} = (a_n - b_n) + \beta_{n+1} (-1)^{n-1} \overline{b}_n \overline{b}_{n-1} \cdots \overline{b}_2 \left(\overline{a}_1 - \overline{b}_1\right) \overset{(4.1.7)}{=} a_n - b_n - b_{n+1} |\beta_n|^2.$$

This is,

$$a_2 - b_2(1 - |\beta_1|^2) = \beta_1.$$

Since $a_{n+1} = a_n$ and $|\beta_{n-1}| \neq 1$, for $n \geq 2$, then

$$b_{n+1} = \frac{b_n}{1 - |\beta_n|^2} = \frac{b_{n-1}}{(1 - |\beta_{n-1}|^2)(1 - |\beta_n|^2)} = \cdots = \frac{b_2}{\prod_{k=2}^{n}(1 - |\beta_k|^2)}, \quad n \geq 2.$$

□

We are interested in the cases when $\mathcal{V}$ is also a positive definite linear functional. Notice that, aside from the trivial case when $a_1 = b_1$, all of the coherence coefficients are determined from the values

$$a_1,\ b_1, \text{ and } b_2 \quad \text{or, equivalently,} \quad a_1,\ b_1, \text{ and } a_2.$$

Not every choice of these parameters will yield a positive definite linear functional $\mathcal{V}$. For instance, if

$$|b_2| = 1 \quad \text{and} \quad |a_1 - b_1| = |\beta_1| = \sqrt{2},$$

then from (4.1.7) we get

$$|b_n| = 1,\ n \geq 3 \quad \text{and} \quad |\beta_n| = \sqrt{2},\ n \geq 2.$$

However, it is possible to choose the values of a_1, b_1, and b_2 in order to get a positive definite linear functional $\mathcal{V}$, or at least a rational spectral transformation of it. We have the following cases.

Proposition 4.1.5. *Let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-coherent pair on the unit circle such that their corresponding monic OPUC satisfy (4.1.1). Let $\mathcal{U}$ be the linear functional associated with the Lebesgue measure. Assume $\mathcal{V}$ is normalized (i.e., $v_0 = 1$). Then,*

(i) *Let $|b_1 - a_1| < 1$. If $a_2 = a_1 - b_1$ (i.e., $b_2 = 0$), then the coherence coefficients satisfy*

$$b_n = 0, \quad a_n = a_1 - b_1, \quad n \geq 2.$$

Furthermore, the monic OPUC with respect to $\mathcal{V}$ and the moments of this linear functional are given by

$$\psi_n(z) = z^{n-1}(z + a_1 - b_1), \quad v_n = (b_1 - a_1)^n, \quad n \geq 1.$$

In other words, $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure with parameter $b_1 - a_1$.

Moreover,

$$b_N = 0 \text{ for some } N \geq 2 \quad \Longrightarrow \quad b_2 = 0.$$

(ii) *If $a_1, b_1, a_2 \in \mathbb{R}$, and either $0 < a_1 - b_1 < a_2 < 1$ or $-1 < a_2 < a_1 - b_1 < 0$ holds, then the Carathéodory function associated with $\mathcal{V}$ is given by*

$$F_{\mathcal{V}} = -\frac{b_1 - a_1}{a_2} F_B(z) + \frac{b_1 - a_1 + a_2}{a_2}, \tag{4.1.13}$$

where $F_B(z)$ is the Carathéodory function associated with the Bernstein-Szegő measure with parameter $-a_2$. In other words, $F_{\mathcal{V}}$ can be obtained by applying a rescaling to the moments of $F_B(z)$, followed by a perturbation of its first moment (i.e, a diagonal perturbation of the corresponding Toeplitz matrix). Thus, the orthogonality measure associated with $\mathcal{V}$ is given by

$$d\mu_1 = -\frac{b_1 - a_1}{a_2} \frac{1 - |a_2|^2}{|1 + a_2 e^{i\theta}|^2} \frac{d\theta}{2\pi} + \frac{b_1 - a_1 + a_2}{a_2} \frac{d\theta}{2\pi}. \tag{4.1.14}$$

(iii) *For any values of a_1, b_1, the value of b_2 can be chosen in such a way that $\mathcal{V}$ is the linear functional associated with a rational spectral transformation of a measure belonging to the Nevai class.*

Proof. **(i)** Note that $a_1 \neq b_1$ because $a_2 \neq 0$. First we will prove that if $b_N = 0$ for some $N \geq 2$, then $b_n = 0$ for $n \geq 2$. Assume that $b_N = 0$ for some $N \geq 2$. From (4.1.8), (4.1.7), and (4.1.6) we get

$$b_n = 0 = \beta_n, \quad \psi_n(z) = z^{n-1}(z + a_2), \quad n \geq N.$$

Besides, other expression for $\psi_N(z)$ is given by the forward Szegő recurrence relation (1.6.2) as follows

$$\psi_N(z) = z\psi_{N-1}(z) + \beta_N \psi^*_{N-1}(z) = z\psi_{N-1}(z),$$

where $\psi_{N-1}(z)$ is given by (4.1.6). Thus, comparing the coefficients of z^{N-1} in both expressions of $\psi_N(z)$ we get that $a_2 = a_2 - b_{N-1}$, and hence $b_{N-1} = 0$. Following the same argument for $b_{N-1}, \ldots, b_2$ we conclude that $b_n = 0$ for $n = 2, \ldots, N-1$, and $a_2 = a_1 - b_1$. Therefore,

$$b_n = 0 = \beta_n, \; n \geq 2, \quad \beta_1 = a_1 - b_1 = a_2, \quad \psi_n(z) = z^{n-1}(z + a_1 - b_1), \quad n \geq 1.$$

As a consequence, from (4.1.8) and (4.1.5), it follows that

$$a_{n+1} = a_1 - b_1, \quad v_n = (b_1 - a_1)^n, \quad n \geq 0.$$

Finally, since $|\beta_1| = |b_1 - a_1| < 1$, then $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure.

(ii) From (4.1.5), the Carathéodory function associated with $\mathcal{V}$ is

$$F_{\mathcal{V}} = 1 + 2\sum_{k=1}^{\infty} \overline{v}_k z^k = 1 + 2\sum_{k=1}^{\infty} (b_1 - a_1)(-a_2)^{k-1} z^k.$$

Since $|a_2| < 1$, then the Bernstein-Szegő polynomials of parameter $-a_2$ have moments $c_n = (-a_2)^n$, are orthogonal with respect to the measure

$$\frac{1-|a_2|^2}{|1+a_2e^{i\theta}|^2}\frac{d\theta}{2\pi},$$

and the associated Carathéodory function is

$$F_B(z) = 1 + 2\sum_{k=1}^{\infty}(-a_2)^k z^k = 1 - 2a_2\sum_{k=1}^{\infty}(-a_2)^{k-1}z^k.$$

Thus,

$$2\sum_{k=1}^{\infty}(-a_2)^{k-1}z^k = \frac{1-F_B(z)}{a_2},$$

and taking into account that

$$\frac{F_{\mathcal{V}}}{b_1-a_1} = \frac{1}{b_1-a_1} + 2\sum_{k=1}^{\infty}(-a_2)^{k-1}z^k,$$

we have

$$\frac{F_{\mathcal{V}}}{b_1-a_1} = \frac{1}{b_1-a_1} + \frac{1-F_B(z)}{a_2} = \frac{b_1-a_1+a_2}{a_2(b_1-a_1)} - \frac{1}{a_2}F_B(z),$$

or, equivalently, (4.1.13).

(iii) From (4.1.8), given $\beta_1 = a_1 - b_1$, we have

$$b_3 = \frac{b_2}{1-|\beta_2|^2} = \frac{b_2}{1-|b_2\beta_1|^2},$$

so we can choose b_2 with small enough norm so that β_2 is sufficiently close to 0. Thus, b_3 will also be close to 0, and since

$$\beta_n = -b_n\beta_{n-1}, \ \ n \geq 2, \quad \text{and} \quad b_n = \frac{b_{n-1}}{1-|\beta_{n-1}|^2}, \ \ n \geq 3,$$

$\{|b_n|\}_{n\geqslant 2}$ will be an increasing sequence and, as a consequence, $\{|\beta_n|\}_{n\geqslant 2}$ will be a decreasing sequence. Besides, b_2 can be chosen so that

$$|b_n| \underset{n\to\infty}{\longrightarrow} b, \quad 0 < b < 1,$$

and, therefore,

$$\prod_{k=2}^{n-1}\left|1-|\beta_k|^2\right| \underset{n\to\infty}{\longrightarrow} \frac{|b_2|}{b}.$$

This shows that

$$\beta_n \underset{n\to\infty}{\longrightarrow} 0,$$

and thus $\{\beta_n\}_{n\geqslant 2}$ defines a Nevai measure μ. Consequently, from Theorem 1.6.2, since $\mathcal{V}$ has $\{\beta_n\}_{n\geqslant 1}$ as Verblunsky coefficients, $\mathcal{V}$ can be expressed as an anti-associated perturbation of order 1 applied to the measure μ. □

Remark 4.1.6. An expression for the monic OPUC with respect to a measure as that obtained in (4.1.14) is given in [29, Section 5] as follows. Let us consider the measure

$$d\mu_1 = d\mu_0 + \hbar\,\frac{d\theta}{2\pi}, \quad \hbar \in \mathbb{R}^+,$$

where μ_0 is the Bernstein-Szegő measure with parameter $-C$ given by

$$d\mu_0 = \frac{1-|C|^2}{|1+Ce^{i\theta}|^2}\frac{d\theta}{2\pi}, \quad C\in\mathbb{C},\ |C|<1,$$

and let $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ be the sequences of monic OPUC associated with μ_0 (see Table 1.6.1) and μ_1, respectively. Then, $\psi_0(z)=1$ and

$$\psi_n(z) = \zeta_n(z)\phi_n(z) + \varsigma_n(z)\phi_n^*(z) = \zeta_n(z)z^{n-1}(z+C) + \varsigma_n(z)\left(1+\overline{C}z\right) \quad n\geq 1,$$

where

$$\zeta_1(z) = 1 - C^2\big(1-|C|^2\big)\left(\hbar + \frac{1}{1-|C|^2}\right),$$
$$\varsigma_1(z) = C\big(1-|C|^2\big)\left(\hbar + \frac{1}{1-|C|^2}\right),$$

and, for $n\geq 2$,

$$\zeta_n(z) = 1 - \frac{|C|^2}{1-|C|^2}\,(n-1)!\,\eta_{n-1,n-1},$$
$$\varsigma_n(z) = \frac{C}{1-|C|^2}\,(n-1)!\left[\eta_{0,n-1} + (z+C)\sum_{j=0}^{n-2}\eta_{j+1,n-1}z^j\right],$$

where $\eta_{i,j}$, $0\leq i,j\leq n-1$, are the entries of matrix $\mathcal{N}_n$ such that

$$\mathcal{N}_n = \left[\eta_{i,j}\right]_{i,j=0}^{n-1} = \left(\frac{1}{\hbar}\mathcal{I}_n + \frac{1}{1-|C|^2}\,\mathcal{C}_n\mathcal{C}_n^T\right)^{-1}\mathcal{D}_n,$$

with $\mathcal{I}_n$ being the identity matrix of order $n \times n$, and

$$\mathcal{C}_n = \begin{bmatrix} 1 & -C & 0 & \cdots & 0 \\ 0 & 1 & -C & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -C \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad \mathcal{D}_n = \operatorname{diag}\left[\frac{1}{0!}, \frac{1}{1!}, \ldots, \frac{1}{(n-1)!}\right].$$

4.1.2 The Bernstein-Szegő Linear Functional

Now, we proceed to analyze the companion measure $\mathcal{V}$ when $\mathcal{U}$ is the Bernstein-Szegő linear functional defined as in Table 1.6.1.

Theorem 4.1.7. *Let $\mathcal{U}$ be the Bernstein-Szegő linear functional and let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-coherent pair on the unit circle given by (4.1.1). Then, the moments of $\mathcal{V}$ are*

$$v_n = (-1)^n \left[(a_1 - b_1) \sum_{k=0}^{n-1} \frac{n+1-k}{n+1} C^k \prod_{j=2}^{n-k} a_j + \frac{1}{n+1} C^n \right] v_0, \quad n \geq 1, \tag{4.1.15}$$

where, by convention, $\prod_{j=k_1}^{k_2} \cdot = 1$ whenever $k_2 < k_1$. The sequence of monic OPUC $\{\psi_n(z)\}_{n\geq 0}$ is given by

$$\psi_0(z) = 1, \quad \psi_1(z) = z + (a_1 - b_1) + \frac{1}{2}C, \quad \text{and for } n \geq 2,$$

$$\begin{aligned} \psi_n(z) = z^n &+ \left[(a_n - b_n) + \frac{n}{n+1} C \right] z^{n-1} - \left[b_n (a_{n-1} - b_{n-1}) - \frac{n-1}{n} C (a_n - b_n) \right] z^{n-2} \\ &+ \sum_{k=0}^{n-3} (-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+3} \left[b_{k+2} (a_{k+1} - b_{k+1}) - \frac{k+1}{k+2} C (a_{k+2} - b_{k+2}) \right] z^k. \end{aligned} \tag{4.1.16}$$

Furthermore, the sequence of Verblunsky coefficients $\{\beta_n = \psi_n(0)\}_{n\geq 1}$ associated with the linear functional $\mathcal{V}$ is

$$\begin{aligned} \beta_1 &= (a_1 - b_1) + \frac{1}{2}C, \\ \beta_2 &= -\left[b_2 (a_1 - b_1) - \frac{1}{2} C (a_2 - b_2) \right] = -\left[b_2 \beta_1 - \frac{1}{2} C a_2 \right], \\ \beta_n &= (-1)^{n-1} b_n b_{n-1} \cdots b_3 \left[b_2 (a_1 - b_1) - \frac{1}{2} C (a_2 - b_2) \right] = -b_n \beta_{n-1}, \quad n \geq 3, \end{aligned} \tag{4.1.17}$$

with $|\beta_n| \neq 1$, $n \geq 1$. The coherence coefficients satisfy

$$a_n + b_n \left[|\beta_{n-1}|^2 - 1 \right] = -\frac{n}{n+1} C + \beta_1 + \frac{1}{2} C a_2 \overline{\beta}_1 - \sum_{k=2}^{n-1} b_k |\beta_{k-1}|^2, \quad n \geq 2. \tag{4.1.18}$$

Proof. Since

$$\phi_n(z) = z^n + Cz^{n-1}, \quad n \geq 1,$$

then

$$\phi_n^{[1]}(z) = z^n + \frac{n}{n+1}Cz^{n-1}, \quad n \geq 0.$$

Hence, from (4.1.4), we get

$$\begin{aligned}(-1)^n(a_1-b_1)\prod_{j=2}^{n} a_j\, v_0 &= (-1)^n(a_1-b_1)\prod_{j=2}^{n} a_j\, \langle \mathcal{V}, 1\rangle \\ &\overset{(4.1.4)}{=} \left\langle \mathcal{V}, \phi_n^{[1]}(z)\right\rangle = v_n + \frac{n}{n+1}Cv_{n-1}, \quad n \geq 1,\end{aligned}$$

this is,

$$v_n = -\frac{n}{n+1}Cv_{n-1} + (-1)^n(a_1-b_1)\prod_{j=2}^{n} a_j\, v_0, \quad n \geq 1, \tag{4.1.19}$$

where, by convention, $\prod_{j=k_1}^{k_2} \cdot = 1$ when $k_2 < k_1$. From the previous equality and by induction on n, we will prove that the moments of $\mathcal{V}$ are given by (4.1.15). Indeed, for $n = 1$, we have

$$v_1 \overset{(4.1.19)}{=} -\left[(a_1-b_1) + \frac{C}{2}\right] v_0.$$

If we assume that (4.1.15) holds for n, then

$$\begin{aligned}v_{n+1} &\underset{(4.1.15)}{\overset{(4.1.19)}{=}} -\frac{n+1}{n+2}C(-1)^n\left[(a_1-b_1)\sum_{k=0}^{n-1}\frac{n+1-k}{n+1}C^k\prod_{j=2}^{n-k} a_j + \frac{1}{n+1}C^n\right] v_0 \\ &\qquad + (-1)^{n+1}(a_1-b_1)\prod_{j=2}^{n+1} a_j\, v_0 \\ &= (-1)^{n+1}\left[(a_1-b_1)\sum_{s=0}^{n}\frac{n+2-s}{n+2}C^s\prod_{j=2}^{n+1-s} a_j + \frac{1}{n+2}C^{n+1}\right] v_0,\end{aligned}$$

which concludes the induction.

From (4.1.3), for $n \geq 1$, we get

$$\begin{aligned}\psi_n(z) &\overset{(4.1.3)}{=} z^n + \frac{n}{n+1}Cz^{n-1} + (a_n-b_n)\left(z^{n-1} + \frac{n-1}{n}Cz^{n-2}\right) \\ &\quad + \sum_{k=0}^{n-2}(-1)^{n-(k+1)}b_nb_{n-1}\cdots b_{k+2}(a_{k+1}-b_{k+1})\left(z^k + \frac{k}{k+1}Cz^{k-1}\right)\end{aligned}$$

$$= z^n + \left[(a_n - b_n) + \frac{n}{n+1}C\right] z^{n-1} - \left[b_n(a_{n-1} - b_{n-1}) - (a_n - b_n)\frac{n-1}{n}C\right] z^{n-2}$$
$$+ \sum_{k=0}^{n-3}(-1)^{n-(k+1)} b_n b_{n-1}\cdots b_{k+3}\left[b_{k+2}(a_{k+1} - b_{k+1}) - (a_{k+2} - b_{k+2})\frac{k+1}{k+2}C\right] z^k,$$

which is (4.1.16) for $n \geq 2$, and $\psi_1(z) = z + (a_1 - b_1) + \frac{1}{2}C$. As a consequence, (4.1.17) holds. Furthermore, since $\{\psi_n(z)\}_{n\geq 0}$ is a sequence of monic OPUC, then $|\beta_n| \neq 1$, $n \geq 1$.

On the other hand, the reversed polynomials of $\{\psi_n(z)\}_{n\geq 0}$ are

$$\psi_1^*(z) = 1 + \left[(\overline{a}_1 - \overline{b}_1) + \frac{1}{2}\overline{C}\right] z$$

and, for $n \geq 2$,

$$\psi_n^*(z) \overset{(4.1.16)}{=} 1 + \left[(\overline{a}_n - \overline{b}_n) + \frac{\overline{C}n}{n+1}\right] z - \left[\overline{b}_n\left(\overline{a}_{n-1} - \overline{b}_{n-1}\right) - \frac{n-1}{n}\overline{C}(\overline{a}_n - \overline{b}_n)\right] z^2$$
$$+ \sum_{j=0}^{n-3}(-1)^{n-j-1}\overline{b}_n\overline{b}_{n-1}\cdots\overline{b}_{j+3}\left[\overline{b}_{j+2}\left(\overline{a}_{j+1} - \overline{b}_{j+1}\right) - \frac{j+1}{j+2}\overline{C}\left(\overline{a}_{j+2} - \overline{b}_{j+2}\right)\right] z^{n-j}$$
$$= 1 + \left[(\overline{a}_n - \overline{b}_n) + \frac{\overline{C}n}{n+1}\right] z - \left[\overline{b}_n\left(\overline{a}_{n-1} - \overline{b}_{n-1}\right) - \frac{n-1}{n}\overline{C}(\overline{a}_n - \overline{b}_n)\right] z^2$$
$$+ \sum_{k=3}^{n}(-1)^{k-1}\overline{b}_n\overline{b}_{n-1}\cdots\overline{b}_{n-k+3}$$
$$\left[\overline{b}_{n-k+2}\left(\overline{a}_{n-k+1} - \overline{b}_{n-k+1}\right) - \frac{n-k+1}{n-k+2}\overline{C}\left(\overline{a}_{n-k+2} - \overline{b}_{n-k+2}\right)\right] z^k. \tag{4.1.20}$$

Since $\{\psi_n(z)\}_{n\geq 0}$ is a sequence of monic OPUC, from the forward Szegő relation we get $\psi_0(z) = 1$, $\psi_1 = z + \beta_1$,

$$\begin{aligned}\psi_2(z) &= z\psi_1(z) + \beta_2\psi_1^*(z)\\ &= z\left\{z + (a_1 - b_1) + \frac{1}{2}C\right\} + \beta_2\left\{1 + \left[(\overline{a}_1 - \overline{b}_1) + \frac{1}{2}\overline{C}\right] z\right\}\\ &= z^2 + \left\{(a_1 - b_1) + \frac{1}{2}C + \beta_2\left[(\overline{a}_1 - \overline{b}_1) + \frac{1}{2}\overline{C}\right]\right\} z + \beta_2,\end{aligned} \tag{4.1.21}$$

$$\psi_3(z) = z\psi_2(z) + \beta_3\psi_2^*(z)$$
$$\underset{(4.1.16)}{\overset{(4.1.20)}{=}} z\left\{z^2 + \left[(a_2 - b_2) + \frac{2}{3}C\right] z - \left[b_2\left(a_1 - b_1\right) - \frac{1}{2}C(a_2 - b_2)\right]\right\}$$

$$
\begin{aligned}
&+ \beta_3 \left\{ 1 + \left[(\overline{a}_2 - \overline{b}_2) + \frac{2}{3}\overline{C} \right] z - \left[\overline{b}_2 \left(\overline{a}_1 - \overline{b}_1 \right) - \frac{1}{2}\overline{C}(\overline{a}_2 - \overline{b}_2) \right] z^2 \right\} \\
= {} & z^3 + \left\{ \left[(a_2 - b_2) + \frac{2}{3}C \right] - \beta_3 \left[\overline{b}_2 \left(\overline{a}_1 - \overline{b}_1 \right) - \frac{1}{2}\overline{C}(\overline{a}_2 - \overline{b}_2) \right] \right\} z^2 \\
&+ \left\{ \beta_3 \left[(\overline{a}_2 - \overline{b}_2) + \frac{2}{3}\overline{C} \right] - \left[b_2 \left(a_1 - b_1 \right) - \frac{1}{2}C(a_2 - b_2) \right] \right\} z + \beta_3,
\end{aligned} \tag{4.1.22}
$$

and for $n \geq 4$,

$$
\begin{aligned}
\psi_n(z) &= z\psi_{n-1}(z) + \beta_n \psi_{n-1}^*(z) \\
&\overset{(4.1.20)}{\underset{(4.1.16)}{=}} z \Bigg\{ z^{n-1} + \left[(a_{n-1} - b_{n-1}) + \frac{C(n-1)}{n} \right] z^{n-2} \\
&\qquad - \left[b_{n-1} \left(a_{n-2} - b_{n-2} \right) - \frac{n-2}{n-1} C(a_{n-1} - b_{n-1}) \right] z^{n-3} \\
&\quad + \sum_{j=0}^{n-4} (-1)^{n-j} b_{n-1} b_{n-2} \cdots b_{j+3} \left[b_{j+2} \left(a_{j+1} - b_{j+1} \right) - \frac{j+1}{j+2} C \left(a_{j+2} - b_{j+2} \right) \right] z^j \Bigg\} \\
&\quad + \beta_n \Bigg\{ 1 + \left[(\overline{a}_{n-1} - \overline{b}_{n-1}) + \frac{\overline{C}(n-1)}{n} \right] z \\
&\quad - \left[\overline{b}_{n-1} \left(\overline{a}_{n-2} - \overline{b}_{n-2} \right) - \frac{n-2}{n-1} \overline{C}(\overline{a}_{n-1} - \overline{b}_{n-1}) \right] z^2 + \sum_{k=3}^{n-1} (-1)^{k-1} \overline{b}_{n-1} \overline{b}_{n-2} \cdots \\
&\qquad \overline{b}_{n-k+2} \left[\overline{b}_{n-k+1} \left(\overline{a}_{n-k} - \overline{b}_{n-k} \right) - \frac{n-k}{n-k+1} \overline{C} \left(\overline{a}_{n-k+1} - \overline{b}_{n-k+1} \right) \right] z^k \Bigg\} \\
&= z^n + \Bigg\{ \left[(a_{n-1} - b_{n-1}) + \frac{n-1}{n} C \right] \\
&\qquad + (-1)^n \beta_n \overline{b}_{n-1} \overline{b}_{n-2} \cdots \overline{b}_3 \left[\overline{b}_2 \left(\overline{a}_1 - \overline{b}_1 \right) - \frac{1}{2} \overline{C} \left(\overline{a}_2 - \overline{b}_2 \right) \right] \Bigg\} z^{n-1} \\
&\quad + \Bigg\{ - \left[b_{n-1} \left(a_{n-2} - b_{n-2} \right) - \frac{n-2}{n-1} C(a_{n-1} - b_{n-1}) \right] \\
&\qquad + (-1)^{n-1} \beta_n \overline{b}_{n-1} \overline{b}_{n-2} \cdots \overline{b}_4 \left[\overline{b}_3 \left(\overline{a}_2 - \overline{b}_2 \right) - \frac{2}{3} \overline{C} \left(\overline{a}_3 - \overline{b}_3 \right) \right] \Bigg\} z^{n-2} \\
&\quad + \sum_{k=3}^{n-3} (-1)^{k-1} \Bigg\{ (-1)^n b_{n-1} b_{n-2} \cdots b_{k+2} \left[b_{k+1} \left(a_k - b_k \right) - \frac{k}{k+1} C \left(a_{k+1} - b_{k+1} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_n\overline{b}_{n-1}\cdots\overline{b}_{n-k+2}\left[\overline{b}_{n-k+1}\left(\overline{a}_{n-k}-\overline{b}_{n-k}\right)-\frac{n-k}{n-k+1}\overline{C}\left(\overline{a}_{n-k+1}-\overline{b}_{n-k+1}\right)\right]\Bigg\}z^k\\
&+\Bigg\{(-1)^{n-1}b_{n-1}b_{n-2}\cdots b_4\left[b_3\left(a_2-b_2\right)-\frac{2}{3}C\left(a_3-b_3\right)\right]\\
&\qquad\qquad -\beta_n\left[\overline{b}_{n-1}\left(\overline{a}_{n-2}-\overline{b}_{n-2}\right)-\frac{n-2}{n-1}\overline{C}(\overline{a}_{n-1}-\overline{b}_{n-1})\right]\Bigg\}z^2\\
&+\Bigg\{(-1)^{n}b_{n-1}b_{n-2}\cdots b_3\left[b_2\left(a_1-b_1\right)-\frac{1}{2}C\left(a_2-b_2\right)\right]\\
&\qquad\qquad +\beta_n\left[(\overline{a}_{n-1}-\overline{b}_{n-1})+\frac{n-1}{n}\overline{C}\right]\Bigg\}z+\beta_n.
\end{aligned}
\tag{4.1.23}
$$

Then, if we compare the coefficients of the expressions of $\psi_n(z)$, $n\geq 2$, given by (4.1.21)-(4.1.23), and (4.1.16), it follows that the coefficient of z for $\psi_2(z)$ is

$$
\begin{aligned}
(a_2-b_2)+\frac{2}{3}C&=(a_1-b_1)+\frac{1}{2}C-\left[b_2\left(a_1-b_1\right)-\frac{1}{2}C(a_2-b_2)\right]\left[(\overline{a}_1-\overline{b}_1)+\frac{1}{2}\overline{C}\right]\\
&\overset{(4.1.17)}{=}\beta_1-\left[b_2\beta_1-\frac{1}{2}Ca_2\right]\overline{\beta}_1=\beta_1+\frac{1}{2}Ca_2\overline{\beta}_1-b_2|\beta_1|^2,
\end{aligned}
\tag{4.1.24}
$$

which is (4.1.18) for $n=2$.

The coefficient of z^2 for $\psi_3(z)$ is

$$
\begin{aligned}
(a_3-b_3)+\frac{3}{4}C&=\left[(a_2-b_2)+\frac{2}{3}C\right]-\beta_3\left[\overline{b}_2\left(\overline{a}_1-\overline{b}_1\right)-\frac{1}{2}\overline{C}(\overline{a}_2-\overline{b}_2)\right]\\
&\overset{(4.1.17)}{=}(a_2-b_2)+\frac{2}{3}C-b_3|\beta_2|^2,
\end{aligned}
\tag{4.1.25}
$$

and for $n\geq 4$, the coefficient of z^{n-1} for $\psi_n(z)$ is

$$
\begin{aligned}
(a_n-b_n)+\frac{n}{n+1}C&=\left[(a_{n-1}-b_{n-1})+\frac{n-1}{n}C\right]\\
&\qquad+(-1)^n\beta_n\overline{b}_{n-1}\overline{b}_{n-2}\cdots\overline{b}_3\left[\overline{b}_2\left(\overline{a}_1-\overline{b}_1\right)-\frac{1}{2}\overline{C}\left(\overline{a}_2-\overline{b}_2\right)\right]\\
&\overset{(4.1.17)}{=}\left[(a_{n-1}-b_{n-1})+\frac{n-1}{n}C\right]-b_n|\beta_{n-1}|^2\\
&=\left[(a_{n-2}-b_{n-2})+\frac{n-2}{n-1}C\right]-b_{n-1}|\beta_{n-2}|^2-b_n|\beta_{n-1}|^2
\end{aligned}
$$

$$
\begin{aligned}
&\vdots\\
&= \left[(a_3-b_3)+\frac{3}{4}C\right]-\sum_{k=4}^{n} b_k|\beta_{k-1}|^2\\
&\overset{(4.1.25)}{=} (a_2-b_2)+\frac{2}{3}C-\sum_{k=3}^{n} b_k|\beta_{k-1}|^2.
\end{aligned}
$$

Hence, for $n \geq 2$,

$$
\begin{aligned}
(a_n-b_n)+\frac{n}{n+1}C &= (a_2-b_2)+\frac{2}{3}C-\sum_{k=3}^{n} b_k|\beta_{k-1}|^2 \qquad (4.1.26)\\
&\overset{(4.1.24)}{=} \beta_1+\frac{1}{2}Ca_2\overline{\beta}_1-\sum_{k=2}^{n} b_k|\beta_{k-1}|^2,
\end{aligned}
$$

which is (4.1.18). □

As in the previous Subsection 4.1.1, we are interested on the situations where $\mathcal{V}$ is also a positive definite linear functional. Notice that now the values of

$$a_1,\ b_1,\ a_2,\ b_2,\ \text{and } b_3$$

determine all other coherence coefficients. We have the following cases.

Proposition 4.1.8. *Let $\mathcal{U}$ be the Bernstein-Szegő linear functional, and let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-coherent pair on the unit circle given by (4.1.1). Then*

(i) *If $a_1 = b_1$, then $C = 0$ and, therefore, $\mathcal{U}$ and $\mathcal{V}$ are Lebesgue linear functionals, and*

$$a_n = b_n, \quad n \geq 1.$$

(ii) *Let $a_1 \neq b_1$.*

(ii.1) *If $\mathcal{V}$ is normalized (i.e., $v_0 = 1$) and $b_N = 0$ for some $N \geq 3$, then $C = 0$, this is, $\mathcal{U}$ is the Lebesgue linear functional. As a consequence, the sequences of coherence coefficients are*

$$b_{n+1} = 0, \quad a_{n+1} = a_1 - b_1, \quad n \geq 1,$$

and the corresponding sequence of monic OPUC with respect to the linear functional $\mathcal{V}$ and its sequence of moments are given by

$$\psi_n(z) = z^{n-1}(z + a_1 - b_1), \quad \text{and} \quad v_n = (b_1 - a_1)^n, \quad n \geq 1.$$

In other words, for $|b_1 - a_1| < 1$, $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure, with parameter $b_1 - a_1$.

(ii.2) *If $\frac{1}{2}Ca_2 = b_2\beta_1$, then the monic OPUC with respect to $\mathcal{V}$ are*

$$\psi_n(z) = z^{n-1}\left(z + a_1 - b_1 + \frac{1}{2}C\right), \quad n \geq 1,$$

this is, for $|b_1 - a_1 - \frac{1}{2}C| < 1$, $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure, with parameter $b_1 - a_1 - \frac{1}{2}C$.

(ii.3) *If $\frac{1}{2}Ca_2 \neq b_2\beta_1$ and $b_n \neq 0$, for $n \geq 3$, then the sequence of coherence coefficients $\{b_n\}_{n\geq 1}$ satifies*

$$b_n = \frac{b_{n-1}}{1 - |\beta_{n-1}|^2} = \frac{b_3}{\prod_{k=3}^{n-1}(1 - |\beta_{k-1}|^2)}, \quad n \geq 4, \tag{4.1.27}$$

and b_3 can be chosen so that $\mathcal{V}$ is the linear functional associated with an anti-associated perturbation of order 2 applied to a measure belonging to the Nevai class (see Theorem 1.6.2).

Proof. **(i)** If we multiply (4.1.16) by z^{-1}, applying $\mathcal{V}$, and using the orthogonality of $\{\psi_n(z)\}_{n\geq 0}$ we get, for $n \geq 2$,

$$\begin{aligned}
0 = v_{n-1} + &\left[(a_n - b_n) + \frac{n}{n+1}C\right] v_{n-2} \\
&- \left[b_n\left(a_{n-1} - b_{n-1}\right) - \frac{n-1}{n}C(a_n - b_n)\right] v_{n-3} \\
&+ \sum_{k=0}^{n-3}(-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+3}\left[b_{k+2}\left(a_{k+1} - b_{k+1}\right) - \frac{k+1}{k+2}C\left(a_{k+2} - b_{k+2}\right)\right] v_{k-1}.
\end{aligned}$$

But this equation for $n+1$ reads

$$\begin{aligned}
0 = v_n + &\left[(a_{n+1} - b_{n+1}) + \frac{n+1}{n+2}C\right] v_{n-1} \\
&- \left[b_{n+1}\left(a_n - b_n\right) - \frac{n}{n+1}C(a_{n+1} - b_{n+1})\right] v_{n-2} \\
&+ \sum_{k=0}^{n-2}(-1)^{n-k} b_{n+1} b_n \cdots b_{k+3}\left[b_{k+2}\left(a_{k+1} - b_{k+1}\right) - \frac{k+1}{k+2}C\left(a_{k+2} - b_{k+2}\right)\right] v_{k-1}.
\end{aligned}$$

If we multiply the first equation by b_{n+1} and adding it to the second equation, we obtain

$$0 = v_n + \left[a_{n+1} + \frac{n+1}{n+2}C\right] v_{n-1} + \frac{n}{n+1}Ca_{n+1}v_{n-2}, \quad n \geq 2. \tag{4.1.28}$$

Then,

$$
\begin{aligned}
0 &\overset{(4.1.28)}{\underset{(4.1.19)}{=}} -\frac{n}{n+1}Cv_{n-1} + (-1)^n(a_1-b_1)\prod_{j=2}^{n} a_j\, v_0 + \left[a_{n+1} + \frac{n+1}{n+2}C\right] v_{n-1} \\
&\quad + \frac{n}{n+1}Ca_{n+1}v_{n-2} \overset{(4.1.19)}{=} (-1)^n(a_1-b_1)\prod_{j=2}^{n} a_j\, v_0 + \frac{n}{n+1}Ca_{n+1}v_{n-2} \\
&\quad + \left[a_{n+1} + \frac{n+1}{n+2}C - \frac{n}{n+1}C\right]\left[-\frac{n-1}{n}Cv_{n-2} + (-1)^{n-1}(a_1-b_1)\prod_{j=2}^{n-1} a_j\, v_0\right],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
0 &= (-1)^{n+1}(a_1-b_1)\prod_{j=2}^{n+1} a_j\, v_0 \left[a_{n+3} - a_{n+2} + \frac{1}{(n+3)(n+4)}C\right] \\
&\quad + \left[\frac{1}{(n+2)(n+3)}a_{n+3} + \frac{n+1}{(n+2)(n+3)(n+4)}C\right] Cv_n, \quad n \geq 0, \quad (4.1.29)
\end{aligned}
$$

where, by convention, $\prod_{j=k_1}^{k_2} \cdot = 1$ whenever $k_2 < k_1$.

On the other hand, the $(1,1)$-coherence relation (4.1.1) yields

$$
z^n + \frac{Cn}{n+1}z^{n-1} + a_n\left[z^{n-1} + \frac{C(n-1)}{n}z^{n-2}\right] = \psi_n(z) + b_n\psi_{n-1}(z), \quad a_n \neq 0,\, n \geq 1,
$$

and, thus, if we apply the linear functional $\mathcal{V}$ to both sides, we get

$$
\begin{aligned}
v_n + \left[a_n + \frac{n}{n+1}C\right] v_{n-1} + a_n\frac{(n-1)}{n}Cv_{n-2} &= 0, \quad n \geq 2, \\
v_1 + \left[a_1 + \frac{1}{2}C\right] v_0 &= b_1 v_0.
\end{aligned}
\qquad (4.1.30)
$$

Hence

$$
\begin{aligned}
0 &\overset{(4.1.28)}{\underset{(4.1.30)}{=}} -\left[a_n + \frac{n}{n+1}C\right] v_{n-1} - a_n\frac{(n-1)}{n}Cv_{n-2} + \left[a_{n+1} + \frac{n+1}{n+2}C\right] v_{n-1} \\
&\quad + \frac{n}{n+1}Ca_{n+1}v_{n-2} = \left[a_{n+1} - a_n + \frac{1}{(n+1)(n+2)}C\right] v_{n-1} \\
&\quad + \left[\frac{n}{n+1}a_{n+1} - \frac{n-1}{n}a_n\right] Cv_{n-2}, \quad n \geq 2. \quad (4.1.31)
\end{aligned}
$$

Therefore, if $a_1 = b_1$, then from (4.1.15), the moments of $\mathcal{V}$ are

$$
v_n = \frac{1}{n+1}(-C)^n v_0, \quad n \geq 0,
$$

and, as a consequence, (4.1.29) becomes

$$0 = (-1)^n \frac{1}{(n+1)(n+2)(n+3)} C^{n+1} \left[a_{n+3} + \frac{n+1}{n+4} C \right] v_0, \quad n \geq 0. \tag{4.1.32}$$

(4.1.31) reads, for $n \geq 2$,

$$0 = (-1)^{n-1} \frac{1}{n(n+1)} C^{n-1} \left[\frac{1}{n+2} C - \frac{1}{n-1} a_{n+1} \right] v_0, \quad n \geq 2. \tag{4.1.33}$$

Then, if $C \neq 0$, from (4.1.32) and (4.1.33) it follows that

$$a_n = -\frac{n-2}{n+1} C, \ n \geq 3, \quad \text{and} \quad a_n = \frac{n-2}{n+1} C, \ n \geq 3,$$

respectively, which is a contradiction. Thus, if $a_1 = b_1$, then $C = 0$, this is $\mathcal{U}$ is the Lebesgue linear functional. In this case, part (i) of the Theorem 4.1.4 holds.

Now, assume $a_1 \neq b_1$.

(ii.1) From part (i) of the Proposition 4.1.5, it suffices to show that $\mathcal{U}$ is the Lebesgue linear functional. Then, let us prove that if $b_N = 0$ for some $N \geq 3$ (and therefore $\beta_N = 0$), then $C = 0$. From (4.1.16) for $n = N$, $N \geq 2$, it follows that

$$\psi_N(z) = z^N + \left(a_N + \frac{N}{N+1} C \right) z^{N-1} + \frac{N-1}{N} C a_N z^{N-2}, \tag{4.1.34}$$

and for $n = N+1$,

$$\begin{aligned}\psi_{N+1}(z) \underset{b_N=0}{\overset{(4.1.16)}{=}} & \ z^{N+1} + \left[(a_{N+1} - b_{N+1}) + \frac{N+1}{N+2} C \right] z^N \\ & - \left[b_{N+1} a_N - \frac{N}{N+1} C (a_{N+1} - b_{N+1}) \right] z^{N-1} - b_{N+1} \frac{N-1}{N} C a_N z^{N-2}.\end{aligned}$$

Hence, for $N \geq 3$, $\beta_{N+1} = \psi_{N+1}(0) = 0$. Furthermore, from the forward Szegő relation we get, for $N \geq 3$,

$$\begin{aligned}\psi_{N+1}(z) &= z\psi_N(z) + \beta_{N+1} \psi_N^*(z) = z\psi_N(z) \\ &\overset{(4.1.34)}{=} z^{N+1} + \left(a_N + \frac{N}{N+1} C \right) z^N + \frac{N-1}{N} C a_N z^{N-1}.\end{aligned}$$

Therefore, if we compare the coefficients of $\psi_{N+1}(z)$ in both expressions, then we obtain, for $N \geq 3$,

$$(a_{N+1} - b_{N+1}) + \frac{N+1}{N+2} C = a_N + \frac{N}{N+1} C, \tag{4.1.35}$$

$$-b_{N+1}a_N + \frac{N}{N+1}C(a_{N+1} - b_{N+1}) = \frac{N-1}{N}Ca_N, \tag{4.1.36}$$

$$b_{N+1}\frac{N-1}{N}Ca_N = 0. \tag{4.1.37}$$

Since $a_N \neq 0$, then from (4.1.37) either $C = 0$ or $b_{N+1} = 0$. If $C = 0$, then from (4.1.36) we get $b_{N+1} = 0$ and, as a consequence, from (4.1.35) we have $a_{N+1} = a_N$. If $b_{N+1} = 0$, then from (4.1.36) it follows that either $C = 0$ (and thus, from (4.1.35), $a_{N+1} = a_N$), or

$$\frac{N}{N+1}a_{N+1} - \frac{N-1}{N}a_N = 0, \quad \text{i.e.,} \quad a_{N+1} = \frac{N^2-1}{N^2}a_N.$$

As a consequence, (4.1.35) yields

$$-\frac{1}{N^2}a_N = \frac{N^2-1}{N^2}a_N - a_N = a_{N+1} - a_N \overset{(4.1.35)}{=} -\frac{1}{(N+1)(N+2)}C,$$

i.e.,

$$C = \frac{(N+1)(N+2)}{N^2}a_N.$$

But since $b_{N+1} = 0$ then, of the same way as for $b_N = 0$, we can conclude that

$$C = \frac{(N+2)(N+3)}{(N+1)^2}a_{N+1},$$

but since $a_{N+1} = \frac{N^2-1}{N^2}a_N$, we obtain

$$\frac{(N+2)(N+3)(N-1)}{(N+1)N^2}\,a_N = C = \frac{(N+1)(N+2)}{N^2}\,a_N,$$

which yields a contradiction. Therefore, $C = 0$.

(ii.2) If $\frac{1}{2}Ca_2 = b_2\beta_1$, then from (4.1.17) we get $\beta_2 = 0$ and, as a consequence, $\beta_n = 0$ for every $n \geq 2$. Therefore, from the forward Szegő relation it follows that

$$\psi_n(z) = z^{n-1}\left(z + \beta_1\right), \quad n \geq 1.$$

(ii.3) If we compare the coefficient of z of the expressions of $\psi_n(z)$, for $n \geq 3$, given by (4.1.22)-(4.1.23) and (4.1.16), it follows that, for $n = 3$, i.e., for $\psi_3(z)$, the coefficient of z is

$$\begin{aligned} -\left[b_3\left(a_2 - b_2\right) - \frac{2}{3}C(a_3 - b_3)\right] &= \beta_3\left[(\overline{a}_2 - \overline{b}_2) + \frac{2}{3}\overline{C}\right] - \left[b_2\left(a_1 - b_1\right) - \frac{C}{2}(a_2 - b_2)\right] \\ &\overset{(4.1.17)}{=} \beta_2\left\{1 - b_3\left[(\overline{a}_2 - \overline{b}_2) + \frac{2}{3}\overline{C}\right]\right\}, \end{aligned} \tag{4.1.38}$$

and, for $n \geq 4$ the coefficient of z is

$$(-1)^n b_n b_{n-1} \cdots b_4 \left[b_3 \left(a_2 - b_2\right) - \frac{2}{3} C \left(a_3 - b_3\right) \right] = (-1)^n b_{n-1} b_{n-2} \cdots b_3 \left[b_2 \left(a_1 - b_1\right) - \frac{1}{2} C \left(a_2 - b_2\right) \right] + \beta_n \left[\left(\overline{a}_{n-1} - \overline{b}_{n-1}\right) + \frac{n-1}{n} \overline{C} \right]$$

$$\overset{(4.1.17)}{\underset{(4.1.26)}{=}} (-1)^{n+1} b_{n-1} \cdots b_3 \, \beta_2 \left\{ 1 - b_n \left[\left(\overline{a}_2 - \overline{b}_2\right) + \frac{2}{3} \overline{C} - \sum_{k=3}^{n-1} \overline{b}_k |\beta_{k-1}|^2 \right] \right\}.$$

Thus, from (4.1.38) we get, for $n \geq 4$,

$$b_n b_{n-1} \cdots b_4 \, \beta_2 \left\{ 1 - b_3 \left[\left(\overline{a}_2 - \overline{b}_2\right) + \frac{2}{3} \overline{C} \right] \right\} = b_{n-1} \cdots b_3 \, \beta_2 \left\{ 1 - b_n \left[\left(\overline{a}_2 - \overline{b}_2\right) + \frac{2}{3} \overline{C} - \sum_{k=3}^{n-1} \overline{b}_k |\beta_{k-1}|^2 \right] \right\},$$

this is,

$$\begin{aligned} \beta_2 \left[b_4 - b_3\right] &= b_4 b_3 \, \beta_2 \sum_{k=3}^{3} \overline{b}_k |\beta_{k-1}|^2, \\ b_{n-1} \cdots b_4 \, \beta_2 \left[b_n - b_3\right] &= b_n b_{n-1} \cdots b_3 \, \beta_2 \sum_{k=3}^{n-1} \overline{b}_k |\beta_{k-1}|^2, \quad n \geq 5. \end{aligned} \tag{4.1.39}$$

So, if $\frac{1}{2} C a_2 \neq b_2 \beta_1$, then from (4.1.17) it follows that $\beta_2 \neq 0$, and as a consequence if $b_4, \ldots, b_{n-1}$, $n \geq 5$, are non-zero, then from (4.1.39) we get

$$b_n = \frac{b_3}{1 - b_3 \sum_{k=3}^{n-1} \overline{b}_k |\beta_{k-1}|^2}, \quad n \geq 4. \tag{4.1.40}$$

Besides, from (4.1.17), $|\beta_n| = |b_n \beta_{n-1}|$ for $n \geq 3$, and if $b_3 \neq 0$, then by induction on n we can prove

$$b_n = \frac{b_{n-1}}{1 - |\beta_{n-1}|^2}, \quad n \geq 4,$$

which is (4.1.27), as follows

$$\begin{aligned} b_{n+1} &\overset{(4.1.40)}{=} \frac{1}{\frac{1 - |\beta_3|^2}{b_3} - \sum_{k=4}^{n} \overline{b}_k |\beta_{k-1}|^2} = \frac{1}{\frac{1}{b_4} - \sum_{k=4}^{n} \overline{b}_k |\beta_{k-1}|^2} \\ &= \frac{1}{\frac{1 - |\beta_4|^2}{b_4} - \sum_{k=5}^{n} \overline{b}_k |\beta_{k-1}|^2} = \frac{1}{\frac{1}{b_5} - \sum_{k=5}^{n} \overline{b}_k |\beta_{k-1}|^2} \end{aligned}$$

$$= \cdots = \frac{1}{\frac{1-|\beta_{n-1}|^2}{b_{n-1}} - \sum_{k=n}^{n} \bar{b}_k |\beta_{k-1}|^2} = \frac{1}{\frac{1}{b_n} - \bar{b}_n |\beta_{n-1}|^2} = \frac{b_n}{1-|\beta_n|^2}.$$

Thus, proceeding as in the proof of Proposition 4.1.5, we can choose b_3 with small enough norm so that β_3 is sufficiently close to 0. As a consequence, $\{|b_n|\}_{n\geqslant 3}$ will be an increasing sequence and hence $\{|\beta_n|\}_{n\geqslant 3}$ will be a decreasing sequence. Also, we can choose b_3 such that

$$|b_n| \underset{n\to\infty}{\longrightarrow} b, \quad \text{with} \quad 0 < b < 1.$$

Then,

$$\prod_{k=3}^{n-1} \left|1 - |\beta_k|^2\right| \underset{n\to\infty}{\longrightarrow} \frac{|b_3|}{b},$$

and therefore, since $\{\beta_n\}_{n\geqslant 1}$ are the Verblunsky coefficients of $\mathcal{V}$, this linear functional $\mathcal{V}$ is an anti-associated perturbation of order 2 applied to a Nevai measure μ. $\square$

4.2 Sobolev Orthogonal Polynomials and (M,N)-Coherent Pairs of Order m on the Unit Circle

Let us consider the Sobolev inner product

$$\left\langle p(z), q(z) \right\rangle_\lambda = \int_{\mathbb{T}} p(z)\overline{q}(1/z)d\mu_0(z) + \lambda \int_{\mathbb{T}} p^{(m)}(z)\overline{q^{(m)}}(1/z)d\mu_1(z), \quad \lambda > 0,\, m \in \mathbb{Z}^+,$$

where $p(z)$ and $q(z)$ are polynomials with complex coefficients, and μ_0 and μ_1 are non-trivial probability measures supported on an infinite subset of the unit circle $\mathbb{T}$, which are associated with positive definite Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively, defined on the linear space of Laurent polynomials, and whose sequences of monic OPUC are $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$, respectively. Thus, the Sobolev inner product can also be written as

$$\begin{aligned} \left\langle p(z), q(z) \right\rangle_\lambda &= \left\langle \mathcal{U}, p(z)\overline{q}(1/z) \right\rangle + \lambda \left\langle \mathcal{V}, p^{(m)}(z)\overline{q^{(m)}}(1/z) \right\rangle \\ &= \left\langle p(z), q(z) \right\rangle_{\mathcal{U}} + \lambda \left\langle p^{(m)}(z), q^{(m)}(z) \right\rangle_{\mathcal{V}}, \quad p, q \in \mathbb{P},\, \lambda > 0,\, m \in \mathbb{Z}^+, \end{aligned} \tag{4.2.1}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{U}}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}}$ are the inner products associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Moreover, using this notation, we can also consider the case when $\mathcal{U}$ and $\mathcal{V}$ are regular Hermitian linear functionals.

Finally, the sequence of monic Sobolev OPUC with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ will be denoted by $\{S_n(z;\lambda)\}_{n\geq 0}$.

Every monic OPUC $S_n(z;\lambda)$, $n\geq 0$, can be written as

$$S_n(z;\lambda)=\frac{\begin{vmatrix} w_{0,0} & \cdots & w_{0,n-1} & w_{0,n}\\ \vdots & \ddots & \vdots & \vdots\\ w_{n-1,0} & \cdots & w_{n-1,n-1} & w_{n-1,n}\\ 1 & \cdots & z^{n-1} & z^n\end{vmatrix}}{\det\left([w_{k,j}]_{k,j=0}^{n-1}\right)},\ \ n\geq 1,\quad S_0(z;\lambda)=1,$$

where

$$\begin{aligned} w_{k,j}=\left\langle z^k,z^j\right\rangle_\lambda &=\left\langle z^k,z^j\right\rangle_{\mathcal{U}}+\lambda\left\langle\left(z^k\right)^{(m)},\left(z^j\right)^{(m)}\right\rangle_{\mathcal{V}}\\ &=u_{k-j}+\lambda(k-m+1)_m(j-m+1)_m v_{(k-m)-(j-m)},\quad k,j\geq 0.\end{aligned}$$

Thus, every coefficient of $S_n(z;\lambda)$ is a rational function of λ such that its numerator and denominator are polynomials of the same degree. Hence, there exist the monic limit polynomials

$$T_n(z)=\lim_{\lambda\longrightarrow\infty}S_n(z;\lambda)\,,\quad n\geq 0. \tag{4.2.2}$$

In this way, for $j<\min\{n,m\}$,

$$\left\langle T_n(z),z^j\right\rangle_{\mathcal{U}}=\lim_{\lambda\longrightarrow\infty}\left[\left\langle S_n(z;\lambda),z^j\right\rangle_\lambda-\lambda\left\langle S_n^{(m)}(z;\lambda),\left(z^j\right)^{(m)}\right\rangle_{\mathcal{V}}\right]=0,$$

and for $j<n$,

$$\left\langle T_n^{(m)}(z),\left(z^j\right)^{(m)}\right\rangle_{\mathcal{V}}=\lim_{\lambda\longrightarrow\infty}\frac{1}{\lambda}\left[0-\left\langle S_n(z;\lambda),z^j\right\rangle_{\mathcal{U}}\right]=0.$$

Therefore,

$$\left\langle T_n(z),z^j\right\rangle_{\mathcal{U}}=0,\ j<\min\{n,m\},\quad\text{and}\quad\left\langle T_n^{(m)}(z),\,z^j\right\rangle_{\mathcal{V}}=0,\ j<n-m. \tag{4.2.3}$$

As a consequence,

$$T_n(z)=\sum_{j=0}^{n}\frac{\langle T_n(z),\phi_j(z)\rangle_{\mathcal{U}}}{\langle\phi_j(z),\phi_j(z)\rangle_{\mathcal{U}}}\,\phi_j(z)=\sum_{j=0}^{n-m}\frac{\langle T_n(z),\phi_{j+m}(z)\rangle_{\mathcal{U}}}{\langle\phi_{j+m}(z),\phi_{j+m}(z)\rangle_{\mathcal{U}}}\,\phi_{j+m}(z),\quad n\geq m, \tag{4.2.4}$$

$$\frac{T_{n+m}^{(m)}(z)}{(n+1)_m}=\sum_{j=0}^{n}\frac{\left\langle\frac{T_{n+m}^{(m)}(z)}{(n+1)_m},\psi_j(z)\right\rangle_{\mathcal{V}}}{\langle\psi_j(z),\psi_j(z)\rangle_{\mathcal{V}}}\,\psi_j(z)=\psi_n(z),\quad n\geq 0, \tag{4.2.5}$$

that yields

$$\psi_n(z) = \phi_n^{[m]}(z) + \sum_{j=0}^{n-1} \frac{(j+1)_m}{(n+1)_m} \frac{\langle T_{n+m}(z), \phi_{j+m}(z)\rangle_{\mathcal{U}}}{\langle \phi_{j+m}(z), \phi_{j+m}(z)\rangle_{\mathcal{U}}} \phi_j^{[m]}(z), \quad n \geq 0.$$

From (4.2.3), for $n \geq 0$ we get

$$T_n(z) = \sum_{j=0}^{n} \frac{\langle T_n(z), S_j(z;\lambda)\rangle_\lambda}{\langle S_j(z;\lambda), S_j(z;\lambda)\rangle_\lambda} S_j(z;\lambda) = S_n(z;\lambda) + \sum_{j=m}^{n-1} \frac{\langle T_n(z), S_j(z;\lambda)\rangle_{\mathcal{U}}}{\langle S_j(z;\lambda), S_j(z;\lambda)\rangle_\lambda} S_j(z;\lambda),$$

and together with (4.2.4), for $n \geq m$,

$$S_n(z;\lambda) + \sum_{j=m}^{n-1} \frac{\langle T_n(z), S_j(z;\lambda)\rangle_{\mathcal{U}}}{\langle S_j(z;\lambda), S_j(z;\lambda)\rangle_\lambda} S_j(z;\lambda) = \phi_n(z) + \sum_{j=m}^{n-1} \frac{\langle T_n(z), \phi_j(z)\rangle_{\mathcal{U}}}{\langle \phi_j(z), \phi_j(z)\rangle_{\mathcal{U}}} \phi_j(z).$$

Finally, from (4.2.1), we obtain that $\left\langle \phi_n(z), z^j \right\rangle_\lambda = 0$, for $j < n < m$, and hence $S_n(z;\lambda) = \phi_n(z)$ for $n < m$, (since the uniqueness of the sequence of monic OPUC).

So, we have proved the following result.

Proposition 4.2.1. *Given the Sobolev inner product (4.2.1), it follows that*

$$\begin{gathered}\psi_n(z) = \phi_n^{[m]}(z) + \sum_{j=0}^{n-1} \frac{(j+1)_m}{(n+1)_m} \frac{\langle T_{n+m}(z), \phi_{j+m}(z)\rangle_{\mathcal{U}}}{\langle \phi_{j+m}(z), \phi_{j+m}(z)\rangle_{\mathcal{U}}} \phi_j^{[m]}(z), \quad n \geq 0, \\ S_n(z;\lambda) + \sum_{j=m}^{n-1} \frac{\langle T_n(z), S_j(z;\lambda)\rangle_{\mathcal{U}}}{\langle S_j(z;\lambda), S_j(z;\lambda)\rangle_\lambda} S_j(z;\lambda) = \phi_n(z) + \sum_{j=m}^{n-1} \frac{\langle T_n(z), \phi_j(z)\rangle_{\mathcal{U}}}{\langle \phi_j(z), \phi_j(z)\rangle_{\mathcal{U}}} \phi_j(z), \;\; n \geq m, \\ S_n(z;\lambda) = \phi_n(z), \quad n \leq m,\end{gathered} \tag{4.2.6}$$

where the monic polynomials $T_n(z)$, $n \geq 0$, are given by (4.2.2).

Now, we are interested in the case when the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ form a (M,N)-*coherent pair of order* m *on the unit circle*, which means that their corresponding sequences of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ satisfy the following algebraic relation

$$\phi_n^{[m]}(z) + \sum_{i=1}^{M} a_{i,n}\phi_{n-i}^{[m]}(z) = \psi_n(z) + \sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z), \quad n \geq 0, \tag{4.2.7}$$

where M, N are non-negative integers, and the sequences $\{a_{i,n}\}_{n\geq 0}, \{b_{i,n}\}_{n\geq 0} \subset \mathbb{C}$ are such that $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ for $i > n$.

In this case, what does (4.2.6) becomes? Our aim now is to answer this question.

If we substitute (4.2.5) in (4.2.7) and then we integrate m times the resulting equation, (4.2.7) becomes

$$\frac{\phi_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{M}a_{i,n}\frac{\phi_{n-i+m}(z)}{(n-i+1)_m}=\frac{T_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{N}b_{i,n}\frac{T_{n-i+m}(z)}{(n-i+1)_m}+\sum_{j=0}^{m-1}\kappa_{n,j}z^j, \quad n\geq 0.$$

Consequently, if we apply $\left\langle \cdot, z^k\right\rangle_{\mathcal{U}}$ for $k=0,1,\ldots,m-1$, to both sides of previous equation, and using (4.2.3), it follows that

$$\sum_{j=0}^{m-1}\kappa_{n,j}u_{j-k}=0, \quad k=0,\ldots,m-1,$$

which is a linear system with an unique solution since $\det\left([u_{j-k}]_{k,j=0}^{m-1}\right)\neq 0$. Therefore, $\kappa_{n,j}=0$, $j=0,\ldots,m-1$, $n\geq 0$. Thus,

$$\frac{\phi_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{M}a_{i,n}\frac{\phi_{n-i+m}(z)}{(n-i+1)_m}=\frac{T_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{N}b_{i,n}\frac{T_{n-i+m}(z)}{(n-i+1)_m}, \quad n\geq 0. \tag{4.2.8}$$

Furthermore,

$$\frac{T_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{N}b_{i,n}\frac{T_{n-i+m}(z)}{(n-i+1)_m}=\frac{S_{n+m}(z;\lambda)}{(n+1)_m}+\sum_{j=1}^{n+m}\frac{c_{j,n,\lambda}}{(n+1)_m}S_{n-j+m}(z;\lambda), \quad n\geq 0, \tag{4.2.9}$$

where from (4.2.8) and (4.2.5), the coefficients $c_{j,n,\lambda}$, for $1\leq j\leq n+m$, $n\geq 0$, are given by

$$\langle S_{n-j+m}(z;\lambda),S_{n-j+m}(z;\lambda)\rangle_\lambda\frac{c_{j,n,\lambda}}{(n+1)_m}=$$
$$\sum_{i=1}^{M}\frac{a_{i,n}\langle\phi_{n-i+m}(z),S_{n-j+m}(z;\lambda)\rangle_{\mathcal{U}}}{(n-i+1)_m}+\lambda\sum_{i=1}^{N}b_{i,n}\left\langle\psi_{n-i}(z),S_{n-j+m}^{(m)}(z;\lambda)\right\rangle_{\mathcal{V}},$$

and, as a consequence, $c_{j,n,\lambda}=0$ for $j>i$ or $j>K=\max\{M,N\}$ or $j>n$. So, from (4.2.8) and (4.2.9) we get

$$\phi_{n+m}(z)+\sum_{i=1}^{M}\frac{(n+1)_m}{(n-i+1)_m}a_{i,n}\phi_{n-i+m}(z)=S_{n+m}(z;\lambda)+\sum_{j=1}^{K}c_{j,n,\lambda}S_{n-j+m}(z;\lambda), \quad n\geq 0.$$

Besides, from the previous proposition $S_n(z;\lambda)=\phi_n(z)$ for $n\leq m$.

Conversely, if the previous equation holds, we can apply $\langle \cdot , p(z)\rangle_\lambda$ in both sides of this equation, for any polynomial $p(z)$ of degree at most $n-K+m-1$. As a consequence,

$$\lambda \left\langle \left(\phi^{(m)}_{n+m}(z) + \sum_{i=1}^{M} \frac{(n+1)_m}{(n-i+1)_m} a_{i,n}\phi^{(m)}_{n-i+m}(z) \right), p^{(m)}(z) \right\rangle_{\mathcal{V}} = 0.$$

In other words,

$$\left\langle \left(\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n}\phi^{[m]}_{n-i}(z) \right), q(z) \right\rangle_{\mathcal{V}} = 0, \quad q \in \mathbb{P}_{n-K-1}.$$

Since

$$\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n}\phi^{[m]}_{n-i}(z) = \psi_n(z) + \sum_{j=1}^{n} b_{j,n}\psi_{n-j}(z), \quad n \geq 0,$$

then, $b_{j,n} = 0$, for $n-j \leq n-K-1$, i.e., $j \geq K+1$, which is a (M,K)-coherence relation of order m.

Summarizing, we have the following theorem that generalizes the relation between Sobolev orthogonal polynomials and $(1,0)$-coherent pairs on the unit circle stated in [23].

Theorem 4.2.2. *If $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order m on the unit circle given by (4.2.7), then*

$$\begin{aligned} \phi_{n+m}(z) + \sum_{i=1}^{M} \frac{(n+1)_m}{(n-i+1)_m} a_{i,n}\phi_{n-i+m}(z) &= S_{n+m}(z;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda} S_{n-j+m}(z;\lambda), \ n \geq 0, \\ S_n(z;\lambda) &= \phi_n(z), \quad n \leq m, \end{aligned} \tag{4.2.10}$$

where $K = \max\{M,N\}$, $c_{j,n,\lambda} = 0$ for $n < j \leq K$, and

$$\begin{aligned} c_{j,n,\lambda} = \frac{(n+1)_m}{\langle S_{n-j+m}(z;\lambda), S_{n-j+m}(z;\lambda)\rangle_\lambda} &\left[\sum_{i=j}^{M} \frac{a_{i,n}\langle \phi_{n-i+m}(z), S_{n-j+m}(z;\lambda)\rangle_{\mathcal{U}}}{(n-i+1)_m} \right. \\ &\left. + \lambda \sum_{i=j}^{N} b_{i,n} \left\langle \psi_{n-i}(z), S^{(m)}_{n-j+m}(z;\lambda) \right\rangle_{\mathcal{V}} \right], \quad 1 \leq j \leq K. \end{aligned} \tag{4.2.11}$$

Conversely, if there exist sequences of complex numbers $\{a_{i,n}\}_{n\geq 0}$, $1 \leq i \leq M$, and $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$ such that (4.2.10) holds with $a_{i,n} = 0$ if $n-i+m < 0$, and $c_{j,n,\lambda} = 0$ if $n-j+m < 0$, then

$$\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n}\phi^{[m]}_{n-i}(z) = \psi_n(z) + \sum_{j=1}^{K} b_{j,n}\psi_{n-j}(z), \quad n \geq 0,$$

i.e., $(\mathcal{U},\mathcal{V})$ *is a* (M,K)*-coherent pair of order* m *on the unit circle (proved* $b_{K,n}\neq 0$*,* $n\geq K$*), where* $b_{j,n}=0$ *for* $n<j\leq K$*, and*

$$b_{j,n}=\frac{\left\langle\left(\phi_n^{[m]}(z)+\sum_{i=1}^{M}a_{i,n}\phi_{n-i}^{[m]}(z)\right),\psi_{n-j}(z)\right\rangle_{\mathcal{V}}}{\langle\psi_{n-j}(z),\psi_{n-j}(z)\rangle_{\mathcal{V}}},\quad 1\leq j\leq\min\{K,n\},\ n\geq 0.$$

Remark 4.2.3. (4.2.11) for $j=K$, is

$$c_{K,n,\lambda}=\frac{(n+1)_m}{\langle S_{n-K+m}(z;\lambda),S_{n-K+m}(z;\lambda)\rangle_\lambda}\Bigg[\frac{a_{M,n}}{(n-M+1)_m}$$
$$\langle\phi_{n-M+m}(z),\phi_{n-M+m}(z)\rangle_{\mathcal{U}}\,\delta_{M,K}+\lambda(n-N+1)_m b_{N,n}\,\langle\psi_{n-N}(z),\psi_{n-N}(z)\rangle_{\mathcal{V}}\,\delta_{N,K}\Bigg],$$

for $n\geq K$, from which it is possible to claim, for every $n\geq K$, that

- if $M>N$ and $a_{M,n}\neq 0$, then $c_{K,n,\lambda}\neq 0$,
- if $M<N$ and $b_{N,n}\neq 0$, then $c_{K,n,\lambda}\neq 0$,
- if $M=N(=K)$ and $a_{M,n}b_{N,n}\neq 0$ then,

$$c_{K,n,\lambda}\neq 0\iff$$
$$a_{K,n}\big\langle\phi_{n-K+m}(z),\phi_{n-K+m}(z)\big\rangle_{\mathcal{U}}+\lambda(n-K+1)_m^2 b_{K,n}\big\langle\psi_{n-K}(z),\psi_{n-K}(z)\big\rangle_{\mathcal{V}}\neq 0.$$

Applying Theorem 4.2.2, we obtain the monic Sobolev OPUC $S_n(z;\lambda)$, $n\geq 0$, as well as the coefficients $c_{j,n,\lambda}$, $1\leq j\leq K$, $n\geq 0$, assuming that $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order m on the unit circle. Nevertheless, we can compute the sequences $\{\langle S_n(z;\lambda),S_n(z;\lambda)\rangle_\lambda\}_{n\geq 0}$ and $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1\leq j\leq K$, without knowing the monic Sobolev OPUC $S_n(z;\lambda)$, $n\geq 0$. Of course, if we also want to obtain these Sobolev polynomials, afterwards, we can get them using (4.2.10).

Indeed, let

$$s_n=\langle S_n(z;\lambda),S_n(z;\lambda)\rangle_\lambda,\quad \widetilde{a}_{i,n}=\frac{(n+1)_m}{(n-i+1)_m}\,a_{i,n},\quad \widetilde{b}_{i,n}=(n+1)_m b_{i,n},\quad n\geq 0,\tag{4.2.12}$$

where $\widetilde{a}_{i,n}=\widetilde{b}_{i,n}=0$ for $i>n$, $\widetilde{a}_{0,n}=1$ and $\widetilde{b}_{0,n}=(n+1)_m$, for $n\geq 0$.

Since (4.2.10) and (4.2.11) hold setting $c_{0,n,\lambda}=1$ for $n\geq 0$, from (4.2.7) and (4.2.10), it follows that (4.2.11) becomes, for $n\geq j$ and $0\leq j\leq K$,

$$s_{n-j+m}c_{j,n,\lambda}=\sum_{i=j}^{M}\sum_{\ell=0}^{M}\widetilde{a}_{i,n}\overline{\widetilde{a}}_{\ell,n-j}\,\langle\phi_{n-i+m}(z),\phi_{n-j-\ell+m}(z)\rangle_{\mathcal{U}}$$

$$
\begin{aligned}
&-\sum_{i=j}^{M}\sum_{\ell=1}^{K}\widetilde{a}_{i,n}\overline{c}_{\ell,n-j,\lambda}\left\langle \phi_{n-i+m}(z), S_{n-j-\ell+m}(z;\lambda)\right\rangle_{\mathcal{U}} \\
&+\lambda\sum_{i=j}^{N}\sum_{\ell=0}^{N}\widetilde{b}_{i,n}\overline{\widetilde{b}}_{\ell,n-j}\left\langle \psi_{n-i}(z), \psi_{n-j-\ell}(z)\right\rangle_{\mathcal{V}} \\
&-\lambda\sum_{i=j}^{N}\sum_{\ell=1}^{K}\widetilde{b}_{i,n}\overline{c}_{\ell,n-j,\lambda}\left\langle \psi_{n-i}(z), S^{(m)}_{n-j-\ell+m}(z;\lambda)\right\rangle_{\mathcal{V}},
\end{aligned}
$$

where

$$\left\langle \phi_{n-i+m}(z), S_{n-j-\ell+m}(z;\lambda)\right\rangle_{\mathcal{U}} = 0, \quad \text{and} \quad \left\langle \psi_{n-i}(z), S^{(m)}_{n-j-\ell+m}(z;\lambda)\right\rangle = 0,$$

for $i < j+\ell$ or $j+\ell > K$ $(\geq M, N)$. Consequently, for $n \geq j$ and $0 \leq j \leq K$, we get

$$
\begin{aligned}
s_{n-j+m}c_{j,n,\lambda} = &\sum_{i=j}^{M}\widetilde{a}_{i,n}\overline{\widetilde{a}}_{i-j,n-j}\left\langle \phi_{n-i+m}(z), \phi_{n-i+m}(z)\right\rangle_{\mathcal{U}} \\
&+\lambda\sum_{i=j}^{N}\widetilde{b}_{i,n}\overline{\widetilde{b}}_{i-j,n-j}\left\langle \psi_{n-i}(z), \psi_{n-i}(z)\right\rangle_{\mathcal{V}} \\
&-\sum_{\ell=1}^{K-j}\overline{c}_{\ell,n-j,\lambda}\sum_{i=j+\ell}^{M}\widetilde{a}_{i,n}\left\langle \phi_{n-i+m}(z), S_{n-j-\ell+m}(z;\lambda)\right\rangle_{\mathcal{U}} \\
&-\lambda\sum_{\ell=1}^{K-j}\overline{c}_{\ell,n-j,\lambda}\sum_{i=j+\ell}^{N}\widetilde{b}_{i,n}\left\langle \psi_{n-i}(z), S^{(m)}_{n-j-\ell+m}(z;\lambda)\right\rangle_{\mathcal{V}}.
\end{aligned}
$$

Notice that the sum of the last two terms is equal to $-\sum_{\ell=1}^{K-j}\overline{c}_{\ell,n-j,\lambda}s_{n-j-\ell+m}c_{j+\ell,n,\lambda}$ using (4.2.11). Therefore, substituting n by $n+j$, we have gotten the recurrence relation stated in the following theorem.

Theorem 4.2.4. *If $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order m on the unit circle given by (4.2.7), then*

$$
\begin{aligned}
&s_{n+m}c_{j,n+j,\lambda} = \zeta_{j,n,\lambda} - \sum_{\ell=1}^{K-j}\overline{c}_{\ell,n,\lambda}c_{j+\ell,n+j,\lambda}s_{n-\ell+m}, \quad 0 \leq j \leq K,\ n \geq 0, \\
&s_n = \left\langle \phi_n(z), \phi_n(z)\right\rangle_{\mathcal{U}},\ n < m, \quad c_{j,n,\lambda} = 0,\ n < j \leq K, \quad c_{0,n,\lambda} = 1,\ n \geq 0,
\end{aligned}
\tag{4.2.13}
$$

where

$$\zeta_{j,n,\lambda} = \sum_{i=j}^{M} \widetilde{a}_{i,n+j}\, \overline{\widetilde{a}}_{i-j,n} \, \langle \phi_{n+j-i+m}(z), \phi_{n+j-i+m}(z) \rangle_{\mathcal{U}} \\ + \lambda \sum_{i=j}^{N} \widetilde{b}_{i,n+j}\, \overline{\widetilde{b}}_{i-j,n} \, \langle \psi_{n+j-i}(z), \psi_{n+j-i}(z) \rangle_{\mathcal{V}}\,, \quad 0 \le j \le K,$$

with s_n*,* $\widetilde{a}_{i,n}$ *and* $\widetilde{b}_{i,n}$*,* $n \geq 0$*, given by* (4.2.12)*, and* $K = \max\{M, N\}$*.*

Remark 4.2.5.

- For the recurrence relation appearing in (4.2.13), we can associate the following matrix with $K+1$ rows and infinitely many columns

$$\begin{bmatrix} s_m & s_{m+1} & s_{m+2} & \cdots & & & & \\ 0 & c_{1,1,\lambda} & c_{1,2,\lambda} & c_{1,3,\lambda} & \cdots & & & \\ 0 & 0 & c_{2,2,\lambda} & c_{2,3,\lambda} & c_{2,4,\lambda} & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & c_{K,K,\lambda} & c_{K,K+1,\lambda} & c_{K,K+2,\lambda} & \cdots \end{bmatrix},$$

  which indicates the order for the computation of the sequences $\{s_{m+n}\}_{n\geq 0}$ and $\{c_{j,n+j,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, through its decreasing diagonals.

- For $j = 0$, (4.2.13) becomes the following non-homogeneous linear difference equation of order K satisfied by $\{s_n\}_{n\geq 0}$

$$s_{n+m} + \sum_{\ell=1}^{K} |c_{\ell,n,\lambda}|^2 s_{n-\ell+m} = \zeta_{0,n,\lambda}, \quad n \geq 0,$$
$$s_n = \langle \phi_n(z), \phi_n(z) \rangle_{\mathcal{U}}\,, \; n < m, \quad c_{j,n,\lambda} = 0, \; n < j \leq K,$$

  where for $n \geq 0$,

$$\zeta_{0,n,\lambda} = \sum_{i=0}^{M} |\widetilde{a}_{i,n}|^2 \, \langle \phi_{n-i+m}(z), \phi_{n-i+m}(z) \rangle_{\mathcal{U}} + \lambda \sum_{i=0}^{N} |\widetilde{b}_{i,n}|^2 \, \langle \psi_{n-i}(z), \psi_{n-i}(z) \rangle_{\mathcal{V}}\,.$$

The sequences $\{s_n = \langle S_n(z;\lambda), S_n(z;\lambda) \rangle_\lambda\}_{n\geq 0}$ and $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, fulfill some additional properties in the cases when $\mathcal{U}$ and $\mathcal{V}$ constitute a $(1,1)$-coherent or $(1,0)$-coherent pair of order m on the unit circle. This will be the topic in the following two subsections.

4.2.1 $(1,1)$-Coherent Pairs of Order m on the Unit Circle

Let us consider a $(1,1)$-coherent pair of order m on the unit circle $(\mathcal{U},\mathcal{V})$ of regular Hermitian linear functionals such that

$$\phi_n^{[m]}(z)+a_{1,n}\phi_{n-1}^{[m]}(z)=\psi_n(z)+b_{1,n}\psi_{n-1}(z),\quad n\geq 0, \tag{4.2.14}$$

with $a_{1,0}=b_{1,0}=0$. Let us consider the sequence of monic Sobolev OPUC $\{S_n(z;\lambda)\}_{n\geq 0}$ with respect to the inner product (4.2.1). Let

$$s_n=\langle S_n(z;\lambda),S_n(z;\lambda)\rangle_\lambda,\quad n\geq 0.$$

Then

i. From Theorem 4.2.2, the monic Sobolev OPUC $S_n(z;\lambda)$, $n\geq 0$, satisfy

$$\begin{gathered}\phi_{m+n}(z)+\frac{n+m}{n}a_{1,n}\phi_{m+n-1}(z)=S_{m+n}(z;\lambda)+c_{1,n,\lambda}S_{m+n-1}(z;\lambda),\quad n\geq 0,\\ S_n(z;\lambda)=\phi_n(z),\quad n\leq m,\end{gathered}$$

where

$$\begin{aligned}c_{1,n,\lambda}=\frac{1}{s_{m+n-1}}\Bigg[&\frac{n+m}{n}a_{1,n}\langle\phi_{m+n-1}(z),\phi_{m+n-1}(z)\rangle_{\mathcal{U}}\\ &+\lambda(n)_m(n+1)_m b_{1,n}\langle\psi_{n-1}(z),\psi_{n-1}(z)\rangle_{\mathcal{V}}\Bigg].\end{aligned}$$

In particular, $c_{1,0,\lambda}=0$.

ii. From Theorem 4.2.4, the sequences $\{s_n\}_{n\geq 0}$ and $\{c_{1,n,\lambda}\}_{n\geq 0}$ satisfy

$$s_{n+m}c_{1,n+1,\lambda}=\zeta_{1,n,\lambda},\quad\text{and}\quad s_{n+m}=\zeta_{0,n,\lambda}-|c_{1,n,\lambda}|^2 s_{n+m-1},\quad n\geq 0,$$

where

$$\begin{aligned}\zeta_{1,n,\lambda}=&\frac{n+m+1}{n+1}a_{1,n+1}\langle\phi_{m+n}(z),\phi_{m+n}(z)\rangle_{\mathcal{U}}\\ &+\lambda(n+1)_m(n+2)_m b_{1,n+1}\langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}},\\ \zeta_{0,n,\lambda}=&\langle\phi_{n+m}(z),\phi_{n+m}(z)\rangle_{\mathcal{U}}+\frac{(n+m)^2}{n^2}|a_{1,n}|^2\langle\phi_{n+m-1}(z),\phi_{n+m-1}(z)\rangle_{\mathcal{U}}\\ &+\lambda(n+1)_m^2\left[\langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}}+|b_{1,n}|^2\langle\psi_{n-1}(z),\psi_{n-1}(z)\rangle_{\mathcal{V}}\right].\end{aligned} \tag{4.2.15}$$

Therefore, if $\zeta_{1,n,\lambda} \neq 0$ for $n \geq 0$, it follows that the constants s_n and $c_{1,n,\lambda}$, $n \geq 0$ are given by

$$s_{m+n+1} = \zeta_{0,n+1,\lambda} - \frac{|\zeta_{1,n,\lambda}|^2}{\overline{s}_{m+n}}, \quad \frac{\zeta_{1,n,\lambda}}{c_{1,n+1,\lambda}} = \zeta_{0,n,\lambda} - \zeta_{1,n-1,\lambda}\overline{c}_{1,n,\lambda}, \quad n \geq 0, \quad (4.2.16)$$

with initial conditions

$$c_{1,0,\lambda} = 0, \quad s_m = \zeta_{0,0,\lambda}, \quad s_n = \langle \phi_n(z), \phi_n(z) \rangle_{\mathcal{U}}, \ n < m.$$

iii. Rewriting the equations in (4.2.16) we get $c_{1,1,\lambda} = \zeta_{1,0,\lambda}/\zeta_{0,0,\lambda}$, and

$$\overline{c}_{1,n,\lambda} = \frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}} - \frac{\frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}}}{c_{1,n+1,\lambda}}, \ n \geq 1, \quad \overline{s}_{m+n} = \frac{|\zeta_{1,n,\lambda}|^2}{\zeta_{0,n+1,\lambda} - s_{m+n+1}}, \ n \geq 0.$$

As a consequence, when $\zeta_{1,n,\lambda} \neq 0$ for $n \geq 0$, every constant s_{m+n}, $n \geq 0$, and $c_{1,n,\lambda}$, $n \geq 1$, can be represented by continued fractions as

$$\overline{s}_{m+n} = \frac{|\zeta_{1,n,\lambda}|^2}{\left[\zeta_{0,n+1,\lambda}\right.} - \frac{\left.|\zeta_{1,n+1,\lambda}|^2\right]}{\left[\zeta_{0,n+2,\lambda}\right.} - \cdots, \quad n \geq 0,$$
$$\overline{c}_{1,n,\lambda} = \frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}} - \frac{\frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}}}{\left[\frac{\zeta_{0,n+1,\lambda}}{\zeta_{1,n,\lambda}}\right.} - \frac{\left.\frac{\zeta_{1,n+1,\lambda}}{\zeta_{1,n,\lambda}}\right]}{\left[\frac{\zeta_{0,n+2,\lambda}}{\zeta_{1,n+1,\lambda}}\right.} - \cdots, \quad n \geq 1.$$

iv. From the theory of continued fractions, it is possible to define a sequence $\{\varpi_{n,\lambda}\}_{n\geq 0}$ as follows

$$\varpi_{0,\lambda} = 1 \quad \text{and} \quad \varpi_{n+1,\lambda} = s_{m+n}\varpi_{n,\lambda}, \ n \geq 0. \quad (4.2.17)$$

So, when $s_n \in \mathbb{R}$, $n \geq 0$, (for instance, when $\mathcal{U}$ and $\mathcal{V}$ are positive definite Hermitian linear functionals) then the first equation in (4.2.16) becomes

$$\varpi_{n+2,\lambda} = \zeta_{0,n+1,\lambda}\varpi_{n+1,\lambda} - |\zeta_{1,n,\lambda}|^2\varpi_{n,\lambda}, \ n \geq 0, \quad \varpi_{1,\lambda} = \zeta_{0,0,\lambda}, \quad \varpi_{0,\lambda} = 1.$$

As a consequence, since $\zeta_{1,n,\lambda} \neq 0$, $n \geq 0$, let us consider the sequence of monic polynomials $\{\varpi_n(x;\lambda)\}_{n\geq 0}$ such that $\varpi_n(0;\lambda) = \varpi_{n,\lambda}$, $n \geq 0$, satisfying the TTRR

$$\begin{gathered} \varpi_{n+1}(x;\lambda) = (x + \zeta_{0,n,\lambda})\varpi_n(x;\lambda) - |\zeta_{1,n-1,\lambda}|^2\varpi_{n-1}(x;\lambda), \quad n \geq 0, \\ \varpi_0(x;\lambda) = 1, \quad \varpi_{-1}(x;\lambda) = 0. \end{gathered} \quad (4.2.18)$$

From the Favard's Theorem, the sequence of monic polynomials $\{\varpi_n(x;\lambda)\}_{n\geq 0}$ is orthogonal with respect to some regular linear functional on $\mathbb{P}$, which will be positive definite when $\zeta_{0,n,\lambda} \in \mathbb{R}$. Then, this sequence will be orthogonal with respect to some positive Borel measure supported on the real line.

Therefore, the sequences $\{s_n\}_{n\geq 0}$ and $\{c_{1,n,\lambda}\}_{n\geq 0}$ fulfill

$$s_{m+n} = \frac{\varpi_{n+1}(0;\lambda)}{\varpi_n(0;\lambda)}, \quad c_{1,n+1,\lambda} = \zeta_{1,n,\lambda}\frac{\varpi_n(0;\lambda)}{\varpi_{n+1}(0;\lambda)}, \quad n \geq 0.$$

Remark 4.2.6. If $c_{1,n,\lambda} \in \mathbb{R}$ and $\zeta_{1,n,\lambda} \neq 0$ for $n \geq 0$, then the same analysis done in the previous item **iv.** holds for the second equation in (4.2.16). In fact, using

$$\theta_{n+1,\lambda} = \frac{\zeta_{1,n,\lambda}/\zeta_{1,n-1,\lambda}}{c_{1,n+1,\lambda}}\theta_{n,\lambda}, \; n \geq 1, \quad \theta_{1,\lambda} = \frac{\zeta_{1,0,\lambda}}{c_{1,1,\lambda}}\theta_{0,\lambda}, \quad \theta_{0,\lambda} = 1,$$

instead of (4.2.17), we get the existence of a SMOP $\{\theta_n(x;\lambda)\}_{n\geq 0}$, such that $\theta_n(0;\lambda) = \theta_{n,\lambda}$ for $n \geq 0$, which satisfies the TTRR

$$\begin{aligned}\theta_{n+1}(x;\lambda) &= \left(x + \frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}}\right)\theta_n(x;\lambda) - \frac{\zeta_{1,n-1,\lambda}}{\zeta_{1,n-2,\lambda}}\theta_{n-1}(x;\lambda), \quad n \geq 2,\\ \theta_2(x;\lambda) &= \left(x + \frac{\zeta_{0,1,\lambda}}{\zeta_{1,0,\lambda}}\right)\theta_1(x;\lambda) - \zeta_{1,0,\lambda}\theta_0(x;\lambda),\\ \theta_1(x;\lambda) &= x + \zeta_{0,0,\lambda}, \quad \theta_0(x;\lambda) = 1.\end{aligned} \tag{4.2.19}$$

Additionally, if $\zeta_{0,n,\lambda}, \zeta_{1,n,\lambda} \in \mathbb{R}$ and $\zeta_{1,n,\lambda} > 0$ for $n \geq 0$, then $\{\theta_n(x;\lambda)\}_{n\geq 0}$ is orthogonal with respect to some positive Borel measure supported on the real line.

Finally, in this case, we obtain that sequences $\{s_n\}_{n\geq 0}$ and $\{c_{1,n,\lambda}\}_{n\geq 0}$ satisfy $c_{1,1,\lambda} = \zeta_{1,0,\lambda}\frac{\theta_0(0;\lambda)}{\theta_1(0;\lambda)}$ and

$$c_{1,n+1,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}}\frac{\theta_n(0;\lambda)}{\theta_{n+1}(0;\lambda)}, \quad s_{m+n} = \zeta_{1,n-1,\lambda}\frac{\theta_{n+1}(0;\lambda)}{\theta_n(0;\lambda)}, \quad n \geq 1.$$

Example 4.2.7. In Proposition 4.1.5 **(i)**, we proved that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order m of Hermitian linear functionals on the unit circle given by (4.2.14), such that $\mathcal{U}$ is the Lebesgue linear functional, $\mathcal{V}$ is normalized, i.e., $v_0 = 1$, $|b_{1,1} - a_{1,1}| < 1$, and, either $a_{1,2} = a_{1,1} - b_{1,1}$ (i.e., $b_{1,2} = 0$) or $b_{1,N} = 0$ for some $N \geq 2$, then $a_{1,n} = a_{1,1} - b_{1,1}$ as well as $b_{1,n} = 0$ for $n \geq 2$, and $\mathcal{V}$ is the Bernstein-Szegő linear functional with parameter $-a_{1,2} = b_{1,1} - a_{1,1}$. In this way,

$$\begin{aligned}\langle \phi_n(z), \phi_n(z)\rangle_{\mathcal{U}} &= \langle z^n, z^n\rangle_{\mathcal{U}} = \langle \mathcal{U}, 1\rangle = 1, \quad n \geq 0,\\ \langle \psi_n(z), \psi_n(z)\rangle_{\mathcal{V}} &= \left\langle z^n + a_{1,2}z^{n-1}, z^n + a_{1,2}z^{n-1}\right\rangle_{\mathcal{V}} = \left\langle \mathcal{V}, 1 + \bar{a}_{1,2}z + a_{1,2}(1/z) + |a_{1,2}|^2\right\rangle\\ &= 1 - |a_{1,2}|^2, \quad n \geq 1, \quad \text{and} \quad \langle \psi_0(z), \psi_0(z)\rangle_{\mathcal{V}} = 1.\end{aligned}$$

Thus, from (4.2.15), the coefficients $\zeta_{0,n,\lambda}$ and $|\zeta_{1,n,\lambda}|^2$, $n \geq 0$, appearing in the TTRR (4.2.18) satisfied by the associated SMOP $\{\varpi_n(x;\lambda)\}_{n\geq 0}$, are

$$\zeta_{0,0,\lambda} = 1 + \lambda(m!)^2, \;\; \zeta_{0,1,\lambda} = 1 + (m+1)^2|a_{1,1}|^2 + \lambda[(m+1)!]^2\big(1 - |a_{1,2}|^2 + |b_{1,1}|^2\big) \in \mathbb{R},$$
$$\zeta_{0,n,\lambda} = 1 + \frac{(n+m)^2}{n^2}|a_{1,2}|^2 + \lambda(n+1)_m^2\big(1 - |a_{1,2}|^2\big) \in \mathbb{R}, \quad n \geq 2,$$
$$\zeta_{1,0,\lambda} = (m+1)a_{1,1} + \lambda(m!)^2(m+1)b_{1,1}, \quad \zeta_{1,n,\lambda} = \frac{n+m+1}{n+1}\, a_{1,2}, \;\; n \geq 1,$$

where $a_{1,2} = a_{1,1} - b_{1,1}$. Additionally, from previous equations, the coefficients of the TTRR (4.2.19) satisfied by the SMOP $\{\theta_n(x;\lambda)\}_{n\geq 0}$ are

$$\frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}} = \frac{1 + \frac{(n+m)^2}{n^2}|a_{1,2}|^2 + \lambda(n+1)_m^2\big(1 - |a_{1,2}|^2\big)}{\frac{n+m}{n}\, a_{1,2}}, \quad n \geq 2,$$
$$\frac{\zeta_{0,1,\lambda}}{\zeta_{1,0,\lambda}} = \frac{1 + (m+1)^2|a_{1,1}|^2 + \lambda[(m+1)!]^2\big(1 - |a_{1,2}|^2 + |b_{1,1}|^2\big)}{(m+1)a_{1,1} + \lambda(m!)^2(m+1)b_{1,1}}, \quad \text{and,}$$

$$\frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}} = \frac{n(n+m+1)}{(n+1)(n+m)} > 0, \;\; n \geq 2, \quad \frac{\zeta_{1,1,\lambda}}{\zeta_{1,0,\lambda}} = \frac{\frac{m+2}{2}\, a_{1,2}}{(m+1)a_{1,1} + \lambda(m!)^2(m+1)b_{1,1}} \neq 0.$$

4.2.2 $(1,0)$-Coherent Pairs of Order m on the Unit Circle

When the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ form a $(1,0)$-coherent pair of order m on the unit circle, i.e., (4.2.14) reads

$$\phi_n^{[m]}(z) + a_{1,n}\phi_{n-1}^{[m]}(z) = \psi_n(z), \;\; n \geq 0, \quad a_{1,0} = 0, a_{1,n} \neq 0, \; n \geq 1,$$

all the results obtained in the previous subsection hold taking $b_{1,n} = 0$, for $n \geq 0$.

In particular, the coefficients $\zeta_{1,n,\lambda}$ and $\zeta_{0,n,\lambda}$ for $n \geq 0$, become

$$\zeta_{1,n,\lambda} = \frac{n+m+1}{n+1}\, a_{1,n+1}\, \langle \phi_{m+n}(z), \phi_{m+n}(z)\rangle_{\mathcal{U}}\,,$$
$$\zeta_{0,n,\lambda} = \langle \phi_{n+m}(z), \phi_{n+m}(z)\rangle_{\mathcal{U}} + \frac{(n+m)^2}{n^2}|a_{1,n}|^2\, \langle \phi_{n+m-1}(z), \phi_{n+m-1}(z)\rangle_{\mathcal{U}}$$
$$+ \lambda(n+1)_m^2\, \langle \psi_n(z), \psi_n(z)\rangle_{\mathcal{V}}\,,$$

i.e., constants and polynomials in the variable λ of degree 1. In this way, in the item **iv.**, it is possible to show that $\varpi_n(0;\lambda)$ is a polynomial in λ of degree n with leading coefficient $\prod_{j=0}^{n-1}(j+1)_m^2\, \langle \psi_j(z), \psi_j(z)\rangle_{\mathcal{V}}$, $n \geq 1$, using induction on n and (4.2.18). Consequently, since $a_{1,n} \neq 0$ for $n \geq 1$, from (4.2.18), the monic polynomials

$$\widetilde{\varpi}_n(\lambda) = \frac{\varpi_n(0;\lambda)}{\prod_{j=0}^{n-1}(j+1)_m^2\, \langle \psi_j(z), \psi_j(z)\rangle_{\mathcal{V}}}, \quad n \geq 1,$$

are a SMOP in the variable λ because they satisfy the following TTRR

$$\widetilde{\varpi}_{n+1}(\lambda) = (\lambda+\alpha_n)\widetilde{\varpi}_n(\lambda) - \beta_n\widetilde{\varpi}_{n-1}(\lambda), \;\; n\geq 0, \quad \widetilde{\varpi}_0(\lambda)=1, \tag{4.2.20}$$

where

$$\alpha_0 = \frac{\langle\phi_m(z),\phi_m(z)\rangle_{\mathcal{U}}}{(1)_m^2\,\langle\psi_0(z),\psi_0(z)\rangle_{\mathcal{V}}}, \quad \beta_0=0,$$

$$\alpha_n = \frac{\langle\phi_{n+m}(z),\phi_{n+m}(z)\rangle_{\mathcal{U}} + \frac{(n+m)^2}{n^2}|a_{1,n}|^2\,\langle\phi_{n+m-1}(z),\phi_{n+m-1}(z)\rangle_{\mathcal{U}}}{(n+1)_m^2\,\langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}}}, \quad n\geq 1,$$

$$\beta_n = \frac{|a_{1,n}|^2|\,\langle\phi_{m+n-1}(z),\phi_{m+n-1}(z)\rangle_{\mathcal{U}}\,|^2}{(n)_m^4\,\langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}}\,\langle\psi_{n-1}(z),\psi_{n-1}(z)\rangle_{\mathcal{V}}}, \quad n\geq 1.$$

As above, $\{\widetilde{\varpi}_{n+1}(\lambda)\}_{n\geq 0}$ is the SMOP with respect to some positive Borel measure supported on the real line if $\alpha_n\in\mathbb{R}$ and $\beta_{n+1}>0$ for $n\geq 0$ (for instance, when $\mathcal{U}$ and $\mathcal{V}$ are positive definite Hermitian linear functionals).

Finally, the sequences $\{s_n\}_{n\geq 0}$ and $\{c_{1,n,\lambda}\}_{n\geq 0}$ satisfy

$$s_{m+n} = \kappa_n\frac{\widetilde{\varpi}_{n+1}(\lambda)}{\widetilde{\varpi}_n(\lambda)}, \;\; n\geq 0, \quad c_{1,n,\lambda} = \widetilde{\kappa}_n\frac{\widetilde{\varpi}_{n-1}(\lambda)}{\widetilde{\varpi}_n(\lambda)}, \;\; n\geq 1,$$

with

$$\kappa_n = (n+1)_m^2\,\langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}}, \quad \widetilde{\kappa}_n = a_{1,n}\,\frac{(n+1)_m}{(n)_m^3}\frac{\langle\phi_{m+n-1}(z),\phi_{m+n-1}(z)\rangle_{\mathcal{U}}}{\langle\psi_{n-1}(z),\psi_{n-1}(z)\rangle_{\mathcal{V}}}.$$

Example 4.2.8. In a similar way as in Example 4.2.7, taking into account that $b_{1,n}=0$, $n\geq 0$, let us consider a $(1,0)$-coherent pair of order m, $(\mathcal{U},\mathcal{V})$, such that $\mathcal{U}$ is the Lebesgue linear functional and $\mathcal{V}$ is normalized, i.e. $v_0=1$. If $|a_{1,1}|<1$, then $a_{1,n}=a_{1,1}$, $n\geq 2$, and $\mathcal{V}$ is the Bernstein-Szegő linear functional with parameter $-a_{1,1}$. Hence,

$$\langle\phi_n(z),\phi_n(z)\rangle_{\mathcal{U}}=1,\; n\geq 0, \quad \langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}} = 1-|a_{1,1}|^2,\; n\geq 1, \quad \langle\psi_0(z),\psi_0(z)\rangle_{\mathcal{V}}=1.$$

As a consequence,

$$\zeta_{1,n,\lambda} = \frac{n+m+1}{n+1}\,a_{1,1}, \;\; n\geq 0, \qquad \zeta_{0,0,\lambda} = 1+\lambda(m!)^2,$$

$$\zeta_{0,n,\lambda} = 1+\frac{(n+m)^2}{n^2}|a_{1,1}|^2 + \lambda(n+1)_m^2\big(1-|a_{1,1}|^2\big), \quad n\geq 1.$$

Besides,

$$\alpha_0 = \frac{1}{(1)_m^2}, \quad \alpha_n = \frac{1+\frac{(n+m)^2}{n^2}|a_{1,1}|^2}{(n+1)_m^2\big(1-|a_{1,1}|^2\big)}\in\mathbb{R}, \quad n\geq 1,$$

$$\beta_0=0, \quad \beta_1 = \frac{|a_{1,1}|^2}{(1)_m^4\big(1-|a_{1,1}|^2\big)}>0, \quad \beta_n = \frac{|a_{1,1}|^2}{(n)_m^4\big(1-|a_{1,1}|^2\big)^2}>0, \quad n\geq 2,$$

are the coefficients of the TTRR (4.2.20) satisfied by the corresponding SMOP $\{\widetilde{\varpi}_n(\lambda)\}_{n\geq 0}$.

4.3 A Matrix Interpretation of (M,N)-Coherence on the Unit Circle

Let $\mathcal{U}$ and $\mathcal{V}$ be two regular Hermitian linear functionals defined on the linear space of Laurent polynomials, and let $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ be their respective sequences of monic OPUC. $(\mathcal{U},\mathcal{V})$ is said to be a (M,N)-*coherent pair on the unit circle* if

$$\frac{\phi'_{n+1}(z)}{n+1}+\sum_{i=1}^{M}a_{i,n}\frac{\phi'_{n-i+1}(z)}{n-i+1}=\psi_n(z)+\sum_{i=1}^{N}b_{i,n}\psi_{n-i}(z),\quad n\geq 0, \tag{4.3.1}$$

holds, where M,N are fixed non-negative integers, and, $a_{i,n}$ and $b_{i,n}$, $n\geq 0$, are complex numbers such that $a_{M,n}\neq 0$ for $n\geq M$, $b_{N,n}\neq 0$ for $n\geq N$, and $a_{i,n}=b_{i,n}=0$ for $i>n$.

If we consider the infinite vectors

$$\begin{gathered}\Phi(z)=\begin{bmatrix}\phi_0(z), & \phi_1(z), & \cdots\end{bmatrix}^T,\quad \Phi_1(z)=\begin{bmatrix}\phi_1(z), & \phi_2(z), & \cdots\end{bmatrix}^T\\ \Psi(z)=\begin{bmatrix}\psi_0(z), & \psi_1(z), & \cdots\end{bmatrix}^T,\end{gathered} \tag{4.3.2}$$

then it is possible to write in two matrix forms the (M,N)-coherence relation (4.3.1) as follows

$$\mathcal{A}_1\Phi'_1(z)=\mathcal{B}\Psi(z), \tag{4.3.3}$$

$$\mathcal{A}\Phi'(z)=\mathcal{B}\Psi(z), \tag{4.3.4}$$

where $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{B}$ are infinite matrices whose 0th rows (counting the rows from zero) are $[1\ 1\ 0\ \cdots]$, $[1\ 0\ \cdots]$, and $[1\ 0\ \cdots]$, respectively, and their corresponding nth rows, for $n\geq 1$, are

$$\begin{bmatrix}\underbrace{0\ \cdots\ 0}_{n-M+1\text{ zeros}} & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{2,n}}{n-1} & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots\end{bmatrix},$$

$$\begin{array}{c}\uparrow\\ n\text{th position}\\ \downarrow\end{array}$$

$$\begin{bmatrix}\underbrace{0\ \cdots\ 0}_{n-M\text{ zeros}} & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots\end{bmatrix},$$

$$\begin{array}{c}\uparrow\\ n\text{th position}\\ \downarrow\end{array}$$

$$\begin{bmatrix}\underbrace{0\ \cdots\ 0}_{n-N\text{ zeros}} & b_{N,n} & \cdots & b_{1,n} & 1 & 0 & \cdots\end{bmatrix},$$

respectively. Thus, $\mathcal{A}$ is a lower Hessenberg matrix with $M+1$ nonzero diagonals, whose entries of its superdiagonal are $\frac{1}{n+1}$, $n\geq 0$, and the entries of its main diagonal are 1 and $\frac{a_{1,n}}{n}$, $n \geq 1$, and $\mathcal{A}_1$ and $\mathcal{B}$ are nonsingular lower triangular matrices with $M+1$ and $N+1$ nonzero diagonals, respectively, such that the entries of their main diagonals are $\frac{1}{n+1}$, $n \geq 0$, and 1's, respectively.

In this way, we can also consider the multiplication operators by z in terms of the basis $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ given in (1.6.3), this is,

$$z\Phi(z) = \mathcal{H}_\phi \Phi(z) \quad \text{and} \quad z\Psi(z) = \mathcal{H}_\psi \Psi(z), \tag{4.3.5}$$

where $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are the infinite lower Hessenberg matrix associated with $\mathcal{U}$ and $\mathcal{V}$, respectively.

So, in this context, we firstly get a previous general result, and then, as propositions, we state its applications to the coherent pairs theory.

Lemma 4.3.1. *If the sequences of monic OPUC $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ satisfy*

$$\Phi'(z) = \mathcal{M}\Psi(z), \tag{4.3.6}$$

where $\Phi(z)$ and $\Psi(z)$ are give by (4.3.2) and $\mathcal{M}$ is a infinite matrix (such that its 0th row is zero since $\phi_0'(x) = 0$), then

$$\mathcal{H}_\phi^2 \mathcal{M} - 2\mathcal{H}_\phi \mathcal{M} \mathcal{H}_\psi + \mathcal{M}\mathcal{H}_\psi^2 = 0, \tag{4.3.7}$$

where $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are the Hessenberg matrices associated with $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$, respectively.

Proof. We have that

$$\mathcal{M}\mathcal{H}_\psi \Psi(z) + \Phi(z) \underset{(4.3.5)}{\overset{(4.3.6)}{=}} z\Phi'(z) + \Phi(z) = \big(z\Phi(z)\big)' \overset{(4.3.5)}{=} \mathcal{H}_\phi \Phi'(z) \overset{(4.3.6)}{=} \mathcal{H}_\phi \mathcal{M}\Psi(z),$$

or equivalently,

$$\Phi(z) = (\mathcal{H}_\phi \mathcal{M} - \mathcal{M}\mathcal{H}_\psi)\, \Psi(z). \tag{4.3.8}$$

Hence,

$$\mathcal{H}_\phi (\mathcal{H}_\phi \mathcal{M} - \mathcal{M}\mathcal{H}_\psi)\, \Psi(z) \overset{(4.3.8)}{=} \mathcal{H}_\phi \Phi(z) \underset{(4.3.5)}{\overset{(4.3.8)}{=}} (\mathcal{H}_\phi \mathcal{M} - \mathcal{M}\mathcal{H}_\psi)\, \mathcal{H}_\psi \Psi(z),$$

and, as a consequence (4.3.7) holds. □

Proposition 4.3.2. *If $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of regular Hermitian linear functionals on the unit circle given (4.3.3), then*

$$\mathcal{H}_\phi^2 \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} - 2\mathcal{H}_\phi \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathcal{H}_\psi + \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathcal{H}_\psi^2 = 0,$$

where $\mathbf{0} = \begin{bmatrix} 0, & 0, & \cdots \end{bmatrix}$ is the zero row.

Proof. The result is immediate from Lemma 4.3.1 taking as $\mathcal{M}$ the matrix obtained from $\mathcal{A}_1^{-1}\mathcal{B}$ by shifting the matrix one position downward, i.e., adding a zero row to top. $\square$

Proposition 4.3.3. *Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-coherent pair of regular Hermitian linear functionals on the unit circle given by (4.3.4), such that the matrix $\mathcal{A}$ is a nonsingular matrix (e.g. when $M=1$ and $N\geq 0$, we have that $a_{1,n}\neq 0$, $n\geq 1$, and, as a consequence, $\mathcal{A}$ is a nonsingular upper bidiagonal matrix). Let $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ be the similar matrices to the corresponding Hessenberg matrices of $\mathcal{U}$ and $\mathcal{V}$, respectively, given by*

$$\mathcal{M}_\phi = \mathcal{A}\mathcal{H}_\phi\mathcal{A}^{-1} \quad \textit{and} \quad \mathcal{M}_\psi = \mathcal{B}\mathcal{H}_\psi\mathcal{B}^{-1}. \tag{4.3.9}$$

Then,

$$(\mathcal{M}_\phi - \mathcal{M}_\psi)^2 = [\mathcal{M}_\phi, \mathcal{M}_\psi],$$

where $[\mathcal{M}_\phi,\mathcal{M}_\psi]$ is the commutator of the matrices $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ defined by

$$[\mathcal{M}_\phi,\mathcal{M}_\psi] = \mathcal{M}_\phi\mathcal{M}_\psi - \mathcal{M}_\psi\mathcal{M}_\phi.$$

Proof. From (4.3.4) and (4.3.7), it follows that

$$\mathcal{H}_\phi^2\mathcal{A}^{-1}\mathcal{B} - 2\mathcal{H}_\phi\mathcal{A}^{-1}\mathcal{B}\mathcal{H}_\psi + \mathcal{A}^{-1}\mathcal{B}\mathcal{H}_\psi^2 = 0.$$

Therefore, multiplying on the left by $\mathcal{A}$ and on the right by $\mathcal{B}^{-1}$ in both sides of the previous equation and taking into account (4.3.9), the previous equation becomes

$$0 = \mathcal{M}_\phi^2 - 2\mathcal{M}_\phi\mathcal{M}_\psi + \mathcal{M}_\psi^2 = (\mathcal{M}_\phi - \mathcal{M}_\psi)^2 - [\mathcal{M}_\phi,\mathcal{M}_\psi].$$

$\square$

Remark 4.3.4. We have a special case when $\mathcal{U}$ is the Lebesgue linear functional because in this case

$$\phi_n^{[m]}(z) = \frac{\phi_{n+m}^{(m)}(z)}{(n+1)_m} = \frac{\left(z^{n+m}\right)^{(m)}}{(n+1)_m} = z^n = \phi_n(z), \quad m,n\geq 0.$$

So, we can consider a (M,N)-coherent pair of order m, $m\geq 0$, of regular Hermitian linear functionals $(\mathcal{U},\mathcal{V})$ on the unit circle, which is defined by the algebraic relation

$$\phi_n^{[m]}(z) + \sum_{i=1}^{M} a_{i,n}\phi_{n-i}^{[m]}(z) = \psi_n(z) + \sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z), \quad n\geq 0, \tag{4.3.10}$$

where $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ are the sequences of monic OPUC with respect to $\mathcal{U}$ and $\mathcal{V}$, respectively, $M,N,m\in\mathbb{N}\cup\{0\}$, $a_{i,n},b_{i,n}\in\mathbb{C}$, $n\geq 0$, $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, and $a_{i,n}=b_{i,n}=0$ if $i>n$.

When $\mathcal{U}$ is the Lebesgue linear functional, (4.3.10) becomes an expression of (M,N)-coherence of order 0, i.e.,

$$\phi_n(z) + \sum_{i=1}^{M} a_{i,n}\phi_{n-i}(z) = \psi_n(z) + \sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z), \quad n \geq 0,$$

which can be written in a matrix form as

$$\widehat{\mathcal{A}}_1\Phi(z) = \mathcal{B}\Psi(z), \tag{4.3.11}$$

where $\Phi(z)$, $\Psi(z)$ and $\mathcal{B}$ are given as in (4.3.4), and the infinite matrix $\widehat{\mathcal{A}}_1$ is as $\mathcal{B}$, i.e., it is a lower triangular matrix with $M+1$ nonzero diagonals, whose entries of its main diagonal are all 1's and its nth row for $n \geq 1$ (counting the rows from zero), is

$$[\ \underbrace{0 \ \cdots \ 0}_{n-M \text{ zeros}} \ a_{M,n} \ \cdots \ a_{1,n} \ 1 \ 0 \ \cdots \ ].$$

Consequently, $\widehat{\mathcal{A}}_1$ is nonsingular as $\mathcal{B}$.

Hence,

$$\widehat{\mathcal{A}}_1\mathcal{H}_\phi\widehat{\mathcal{A}}_1^{-1}\mathcal{B}\Psi(z) \overset{(4.3.11)}{=} \widehat{\mathcal{A}}_1\mathcal{H}_\phi\Phi(z) \overset{(4.3.5)}{=} z\widehat{\mathcal{A}}_1\Phi(z) \overset{(4.3.11)}{=} z\mathcal{B}\Psi(z) \overset{(4.3.5)}{=} \mathcal{B}\mathcal{H}_\psi\Psi(z),$$

from which we can conclude that

$$\widehat{\mathcal{A}}_1\mathcal{H}_\phi\widehat{\mathcal{A}}_1^{-1} = \widehat{\mathcal{M}}_\phi = \mathcal{M}_\psi = \mathcal{B}\mathcal{H}_\psi\mathcal{B}^{-1},$$

where $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are the Hessenberg matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Therefore, $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are similar matrices.

4.3.1 A Matrix Interpretation of Sobolev Orthogonal Polynomials and (M,N)-Coherence of Order m on the Unit Circle

Let us recall that there is a close relation between Sobolev orthogonal polynomials and (M,N)-coherent pairs of order m on the unit circle, $m \geq 1$. More precisely, in Theorem 4.2.2 we showed that the (M,N)-coherence relation of order m on the unit circle (4.2.7) satisfied by the sequences of monic OPUC associated with the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$, implies the algebraic relation (4.2.10), i.e.,

$$\phi_{n+m}(z) + \sum_{i=1}^{M} \widetilde{a}_{i,n}\phi_{n-i+m}(z) = S_{n+m}(z;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda}S_{n-j+m}(z;\lambda), \quad n \geq 0,$$
$$S_n(z;\lambda) = \phi_n(z), \quad n \leq m,$$

where $K = \max\{M, N\}$, $c_{j,n,\lambda} = 0$ for $n < j \le K$,

$$\widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n}, \quad n \ge 0,$$

$\widetilde{a}_{i,n} = 0$, $i > n$, $\widetilde{a}_{0,n} = 1$, $n \ge 0$, and $S_n(z;\lambda)$, $n \ge 0$, are the monic Sobolev polynomials on the unit circle orthogonal with respect to the inner product (4.2.1)

$$\left\langle p(z), q(z)\right\rangle_\lambda = \left\langle \mathcal{U}, p(z)\overline{q}(1/z)\right\rangle + \lambda \left\langle \mathcal{V}, p^{(m)}(z)\overline{q^{(m)}}(1/z)\right\rangle, \quad p, q \in \mathbb{P},\ \lambda > 0,\ m \in \mathbb{Z}^+.$$

In this way, if we write these algebraic relations as

$$\widetilde{\mathcal{A}}\Phi(z) = \mathcal{C}\mathbf{s}(z;\lambda),$$

where

$$\Phi(z) = \left[\ \phi_0(z),\quad \phi_1(z),\quad \cdots\ \right]^T, \quad \mathbf{s}(z;\lambda) = \left[\ S_0(z;\lambda),\quad S_1(z;\lambda),\quad \cdots\ \right]^T,$$

and

$$\begin{bmatrix} \underbrace{0\ \cdots\ 0}_{n-M+m\text{ zeros}} & \widetilde{a}_{M,n} & \cdots & \widetilde{a}_{1,n} & 1 & 0 & \cdots \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ (n+m)\text{th place} \\ \downarrow \end{array}$$

$$\begin{bmatrix} \underbrace{0\ \cdots\ 0}_{n-K+m\text{ zeros}} & c_{K,n,\lambda} & \cdots & c_{1,n,\lambda} & 1 & 0 & \cdots \end{bmatrix}$$

are the $(n+m)$th rows, respectively, $n \ge 0$, and $[0\ \cdots\ 0\ 1\ 0\ \cdots]$ are the first m rows (counting from zero) of the lower triangular matrices $\widetilde{\mathcal{A}}$ and $\mathcal{C}$ (notice that these matrices have $M+1$ and $K+1$ nonzero diagonals, respectively, and 1's as entries of their main diagonal), then we obtain the following result.

Proposition 4.3.5. *If $(\mathcal{U}, \mathcal{V})$ is a (M,N)-coherent pair of order m on the unit circle of regular Hermitian linear functionals and $m \ge 1$, then the matrix representation of the multiplication operator by z in terms of $\{S_n(z;\lambda)\}_{n\ge 0}$, the basis of monic Sobolev OPUC with respect to the inner product* (4.2.1)*, is given by*

$$z\mathbf{s}(z;\lambda) = \mathcal{H}_{S,\phi,\lambda}\mathbf{s}(z;\lambda),$$

where $\mathcal{H}_{S,\phi,\lambda}$ is a lower Hessenberg matrix similar to the Hessenberg matrix associated with the linear functional $\mathcal{U}$.

Proof. Since

$$z\mathbf{s}(z;\lambda) \overset{(4.3.1)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}z\Phi(z) \overset{(1.6.3)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{H}_{\phi}\Phi(z) \overset{(4.3.1)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{H}_{\phi}\widetilde{\mathcal{A}}^{-1}\mathcal{C}\mathbf{s}(z;\lambda),$$

and if

$$\mathcal{H}_{S,\phi,\lambda} = \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{H}_{\phi}\widetilde{\mathcal{A}}^{-1}\mathcal{C} = \left[\mathcal{C}^{-1}\widetilde{\mathcal{A}}\right]\mathcal{H}_{\phi}\left[\mathcal{C}^{-1}\widetilde{\mathcal{A}}\right]^{-1},$$

then the proof is complete. $\square$

Conclusions and Open Problems

Conclusions

In addition to the original contributions of this dissertation as presented in the Introduction, we can conclude the following

- All the generalizations of the notion of coherence studied until now may be considered as special cases of the notion of (M, N)-coherence of order (m, k) discussed in this dissertation. This is also true for the D_ν-coherence case as well as for the coherence case on the unit circle. In this way, the results obtained in our work generalize those proved for the particular cases.

- An important characterization for (M, N) and (M, N)-D_ν -coherent pairs of order (m, k) was deduced. The linear functionals that constitute a (M, N) (resp. (M, N)-D_ν) -coherent pair of order (m, k) must be semiclassical (resp. D_ν-semiclassical), whenever $m \neq k$, and they are related by a rational factor.

- We have proved an useful algebraic relation between the Sobolev orthogonal polynomials associated with the inner product formed by the linear functionals in either a (M, N)-coherent pair of order m or a (M, N)-D_ν-coherent pair of order m or a (M, N)-coherent pair of order m on the unit circle, and the polynomials orthogonal with respect to the first linear functional of such a coherent pair. Conversely, such an algebraic relation is a sufficient condition for either (M, K)-coherence of order m or (M, K)-D_ν-coherence of order m or (M, K)-coherence of order m on the unit circle, respectively, with $K = \max\{M, N\}$.

- We have built and implemented an efficient algorithm for the computation of the Fourier-Sobolev coefficients of functions of the Sobolev space $W^{m,2}(I, \mu_0, \mu_1)$ when (μ_0, μ_1) is a (M, N)-coherent pair of order m, which does not need the explicit expressions of the Sobolev orthogonal polynomials.

- From a matrix interpretation of the (M, N) and (M, N)-D_ν -coherence relations (resp. (M, N)-coherence relation on the unit circle), we pointed out some interesting results involving the monic Jacobi matrices (resp. Hessenberg matrices) associated with the linear functionals of the coherent pair.
- In the particular cases either of $(1, 0)$ and $(1, 1)$ (resp. $(1, 0)$-D_ν and $(1, 1)$-D_ν) -coherence, or when one of the linear functionals in a (M, N) (resp. (M, N)-D_ν) -coherent pair of order (m, k) is classical (resp. D_ν-classical), it is always possible to obtain additional properties. This is also holds for coherence on the unit circle when considering the Lebesgue and Bernstein-Szegő linear functionals.
- We gave a classification of the $(1, 1)$-coherent pairs on the unit circle when the first linear functional of the coherent pair is either the Lebesgue or the Bernstein-Szegő linear functional.

Open Problems

1. Concerning the analysis of coherent pairs

- To describe all (M, N)-coherent pairs of order (m, k) for fixed values of M, N, m, and k, as the results obtained by H. G. Meijer in [98, 99], and by A. Delgado and F. Marcellán in [32, 33] for $(1, 0)$ and $(1, 1)$ -coherent pairs, respectively.
- To describe all (M, N)-D_ν-coherent pairs of order (m, k) for fixed values of M, N, m, and k, in a similar way to the results of I. Area, E. Godoy, and F. Marcellán in [14, 15, 16, 17] for $(1, 0)$-D_ν-coherent pairs.
- To describe all (M, N)-coherent pairs of order (m, k) on the unit circle for fixed values of M, N, m, and k, taking into account the results showed by A. Branquinho, A. Foulquié Moreno, F. Marcellán, and M. N. Rebocho in [23], by A. Branquinho and M. N. Rebocho in [26], and, by L. Garza, F. Marcellán, and N. C. Pinzón-Cortés in [38] and here, for $(1, 0)$, $(0, 1)$, and $(1, 1)$ -coherent pairs, respectively, when one of the linear functionals is either Lebesgue or Bernstein-Szegő.

2. Concerning analytic properties of Sobolev orthogonal polynomials

- To study asymptotic properties of the Sobolev polynomials $S_n(x; \lambda)$, $S_n(x; \lambda, \nu)$, and $S_n(z; \lambda)$, $n \geq 0$, orthogonal with respect to the inner products

$$\left\langle p(x), r(x) \right\rangle_\lambda = \left\langle \mathcal{U}, p(x)r(x) \right\rangle + \lambda \left\langle \mathcal{V}, p^{(m)}(x)r^{(m)}(x) \right\rangle, \quad \lambda > 0, \; m \in \mathbb{Z}^+,$$

$$\langle p(x), r(x)\rangle_{\lambda,\nu} = \langle \mathcal{U}, p(x)r(x)\rangle + \lambda\Big\langle \mathcal{V}, \big(D_\nu^m p\big)(x)\big(D_\nu^m r\big)(x)\Big\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$
$$\langle p(z), r(z)\rangle_{\lambda} = \langle \mathcal{U}, p(z)\overline{r}(1/z)\rangle + \lambda\Big\langle \mathcal{V}, p^{(m)}(z)\overline{r^{(m)}}(1/z)\Big\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$

respectively, when $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order m, a (M, N)-D_ν-coherent pair of order m, and a (M, N)-coherent pair of order m on the unit circle, respectively.

- To study the zeros of the Sobolev orthogonal polynomials $S_n(x; \lambda)$, $S_n(x; \lambda, \nu)$, and $S_n(z; \lambda)$, $n \geq 0$.
- To analyze the convergence of Fourier series expansions in terms of the Sobolev orthogonal polynomials $S_n(x; \lambda)$, $S_n(x; \lambda, \nu)$, and $S_n(z; \lambda)$, $n \geq 0$.

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